6th Iranian Geometry Olympiad

Contest problems with solutions

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Contest problems with solutions.

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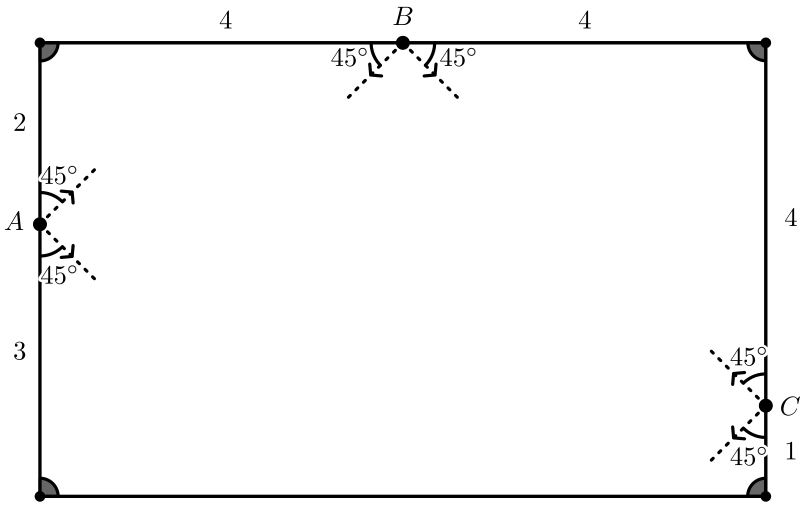
Elementary Level

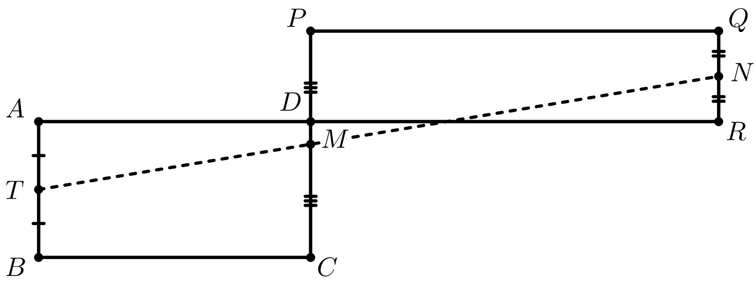
Problems

1) There is a table in the shape of a 8×5 rectangle with four holes on its corners. After shooting a ball from points A, B and C on the shown paths, will the ball fall into any of the holes after 6 reflections? (The ball reflects with the same angle after contacting the table edges.)



(→ p.4)

2) As shown in the figure, there are two rectangles $ABCD$ and $PQRD$ with the same area, and with parallel corresponding edges. Let points N, M and T be the midpoints of segments QR, PC and AB, respectively. Prove that points N, M and T lie on the same line.



(→ p.7)

3) There are $n > 2$ lines on the plane in general position; Meaning any two of them meet, but no three are concurrent. All their intersection points are marked, and then all the lines are removed, but the marked points are remained. It is not known which marked point belongs to which two lines. Is it possible to know which line belongs where, and restore them all?

(→ p.8)

4) Quadrilateral $ABCD$ is given such that

$$\angle DAC = \angle CAB = 60^\circ,$$

and

$$AB = BD - AC.$$

Lines AB and CD intersect each other at point E. Prove that

$$\angle ADB = 2\angle BEC.$$

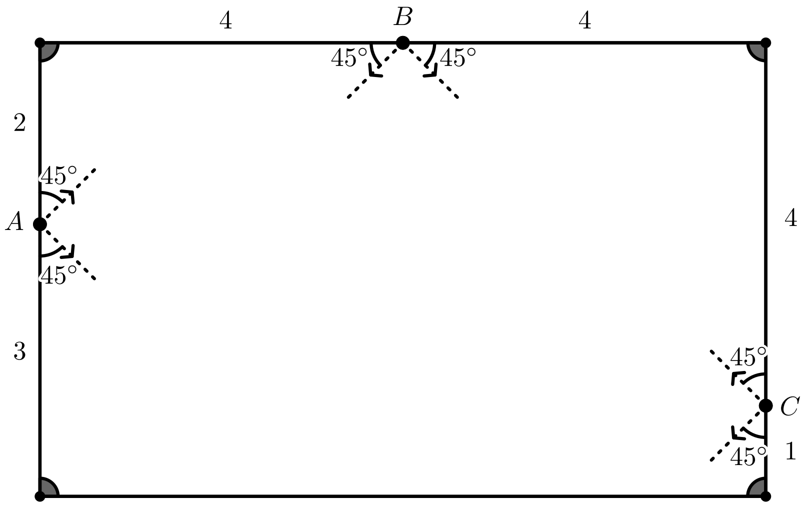
(→ p.9)

5) For a convex polygon (i.e. all angles less than 180°) call a diagonal *bisector* if its bisects both area and perimeter of the polygon. What is the maximum number of bisector diagonals for a convex pentagon?

(→ p.10)

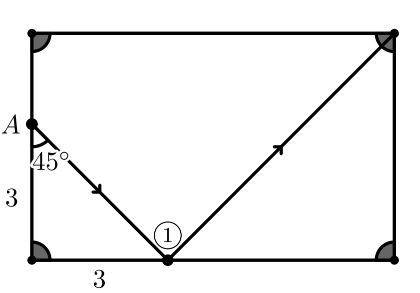
Solutions

1) There is a table in the shape of a 8×5 rectangle with four holes on its corners. After shooting a ball from points A, B and C on the shown paths, will the ball fall into any of the holes after 6 reflections? (The ball reflects with the same angle after contacting the table edges.)

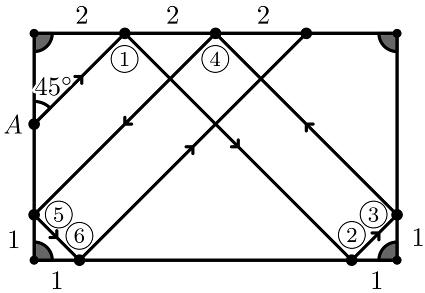


Proposed by Hirad Alipanah

Solution. It's easy to track the trajectory of the ball.
Point A:

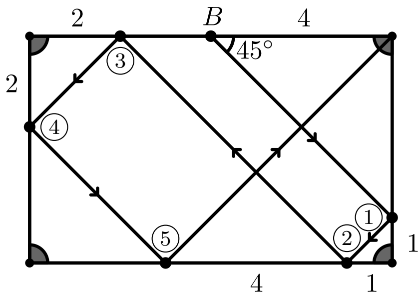


(a) The ball goes through a hole with one reflection.

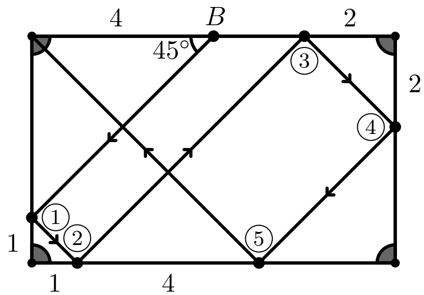


(b) The ball doesn't go through a hole after six reflections.

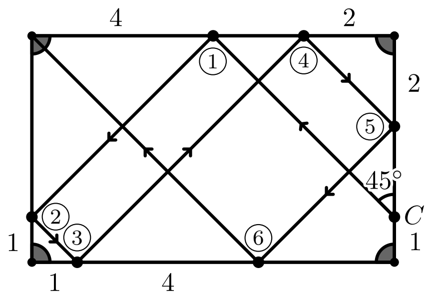
Point B:



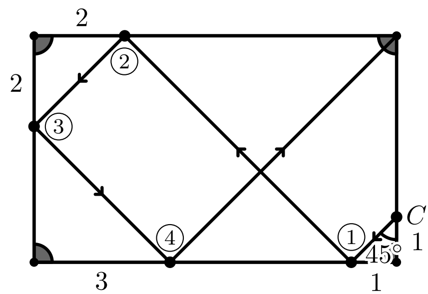
(a) The ball goes through a hole with five reflections.



(b) The ball goes through a hole with five reflections.



(a) The ball goes through a hole with six refelctions.



(b) The ball goes through a hole with four reflections.

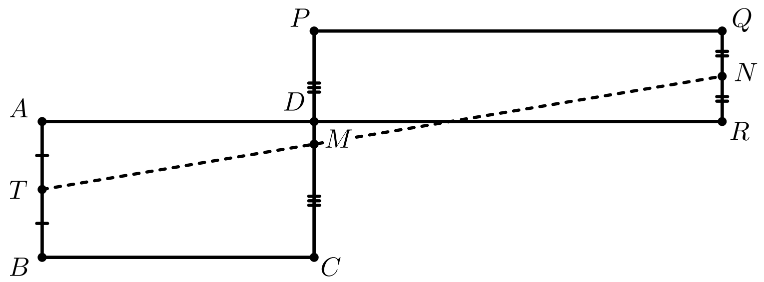
There are two point of views to the problem:

(a) Looking for the trajectories where the ball goes through a hole with at most 6 reflections:
In this case, all cases except A(b) are desired.

(b) Looking for the trajectories where the ball goes through a hole with exactly 6 reflections:
In this case, C(a) is the only answer to the problem.

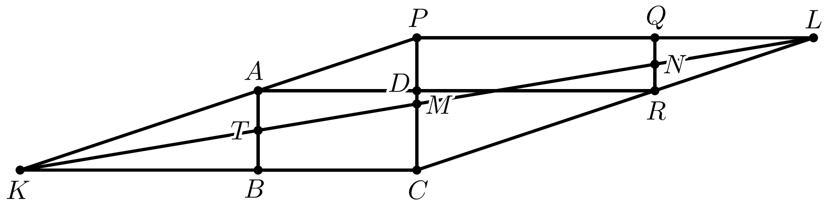
■

2) As shown in the figure, there are two rectangles $ABCD$ and $PQRD$ with the same area, and with parallel corresponding edges. Let points N, M and T be the midpoints of segments QR, PC and AB, respectively. Prove that points N, M and T lie on the same line.



Proposed by Morteza Saghafian

Solution. Let L be the intersection point of PQ and CR, and let K be the intersection point of BC and AP.



Since $QR \parallel PC$, it is deduced that L, N and M are collinear; Similarly, K, T and M are collinear. Therefore it suffices to prove that $PLCK$ is a parallelogram to deduce that K, M and L are collinear, and get the desired result of the problem. Since $PL \parallel CK$, it suffices to show that $PK \parallel CL$, or $PA \parallel CR$. Since the areas of the two rectangles are equal, it is implied that

$$PD \cdot DR = AD \cdot CD \implies \frac{PD}{CD} = \frac{AD}{DR}.$$

Which implies $AP \parallel CR$, and the proof is complete. ■

3) There are $n > 2$ lines on the plane in general position; Meaning any two of them meet, but no three are concurrent. All their intersection points are marked, and then all the lines are removed, but the marked points are remained. It is not known which marked point belongs to which two lines. Is it possible to know which line belongs where, and restore them all?

Proposed by Boris Frenkin - Russia

Answer. Yes, it is.

Solution. Draw the lines which each of them contains $n-1$ marked points, at least. All the original lines are among these lines. Conversely, let some line ℓ contains some $n-1$ marked points. They are points of meet of some pairs of the original lines $(\ell_1, \ell_2), (\ell_3, \ell_4), \ldots, (\ell_{2n-3}, \ell_{2n-2})$. Since $n > 2$, we have $2n - 2 > n$, so ℓ_i coincides with ℓ_j for some $1 \leq i < j \leq 2n - 2$. Then these lines belong to distinct pairs in the above list, and the two corresponding marked points belong to $\ell_i = \ell_j$. But then also $\ell = \ell_i$, and we are done. ■

4) Quadrilateral $ABCD$ is given such that

$$\angle DAC = \angle CAB = 60^\circ,$$

and

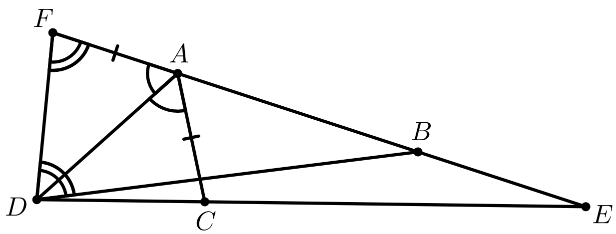
$$AB = BD - AC.$$

Lines AB and CD intersect each other at point E. Prove that

$$\angle ADB = 2\angle BEC.$$

Proposed by Iman Maghsoudi

Solution. Consider point F on ray BA such that $AF = AC$.



Knowing that $AB = BD - AC$, it is implied that $BF = BD$. Therefore

$$\left.\begin{array}{l} AF = AC \\ AD = AD \\ \angle FAD = \angle CAD = 60^\circ \end{array}\right\} \implies \triangle FAD \cong \triangle CAD. \tag{1}$$

Note that

$$\angle BEC = \angle FAD - \angle ADC \overset{(1)}{=} 60^\circ - \angle ADF. \tag{2}$$

On the other hand

$$\begin{aligned} \angle ADB = \angle FDB - \angle ADF &= \angle AFD - \angle ADF \\ &= (120^\circ - \angle ADF) - \angle ADF \\ &= 120^\circ - 2\angle ADF \\ &\overset{(2)}{=} 2\angle BEC. \end{aligned}$$

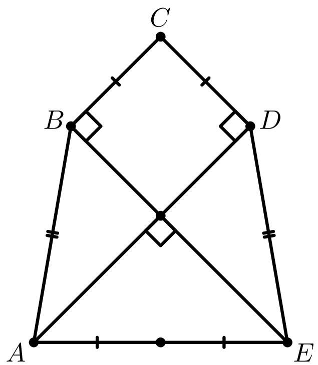
So the claim of the problem is proved. ■

5) For a convex polygon (i.e. all angles less than 180°) call a diagonal *bisector* if its bisects both area and perimeter of the polygon. What is the maximum number of bisector diagonals for a convex pentagon?

Proposed by Morteza Saghafian

Answer. The maximum number of bisector diagonals is 2.

Solution. Note that for each vertex, there is at most one bisector diagonal that passes through it; Therefore there are at most 2 bisector diagonals in the pentagon. The following figure shows an example where the pentagon has two bisector diagonals.



■

Intermediate Level

Problems

1) Two circles ω_1 and ω_2 with centers O_1 and O_2 respectively intersect each other at points A and B, and point O_1 lies on ω_2. Let P be an arbitrary point lying on ω_1. Lines BP, AP and O_1O_2 cut ω_2 for the second time at points X, Y and C, respectively. Prove that quadrilateral $XPYC$ is a parallelogram.

(→ p.13)

2) Find all quadrilaterals $ABCD$ such that all four triangles DAB, CDA, BCD and ABC are similar to one-another.

(→ p.15)

3) Three circles ω_1, ω_2 and ω_3 pass through one common point, say P. The tangent line to ω_1 at P intersects ω_2 and ω_3 for the second time at points $P_{1,2}$ and $P_{1,3}$, respectively. Points $P_{2,1}$, $P_{2,3}$, $P_{3,1}$ and $P_{3,2}$ are similarly defined. Prove that the perpendicular bisector of segments $P_{1,2}P_{1,3}$, $P_{2,1}P_{2,3}$ and $P_{3,1}P_{3,2}$ are concurrent.

(→ p.16)

4) Let $ABCD$ be a parallelogram and let K be a point on line AD such that $BK = AB$. Suppose that P is an arbitrary point on AB, and the perpendicular bisector of PC intersects the circumcircle of triangle APD at points X, Y. Prove that the circumcircle of triangle ABK passes through the orthocenter of triangle AXY.

(→ p.19)

5) Let ABC be a triangle with $\angle A = 60°$. Points E and F are the foot of angle bisectors of vertices B and C respectively. Points P and Q are considered such that quadrilaterals $BFPE$ and $CEQF$ are parallelograms. Prove that $\angle PAQ > 150°$. (Consider the angle PAQ that does not contain side AB of the triangle.)

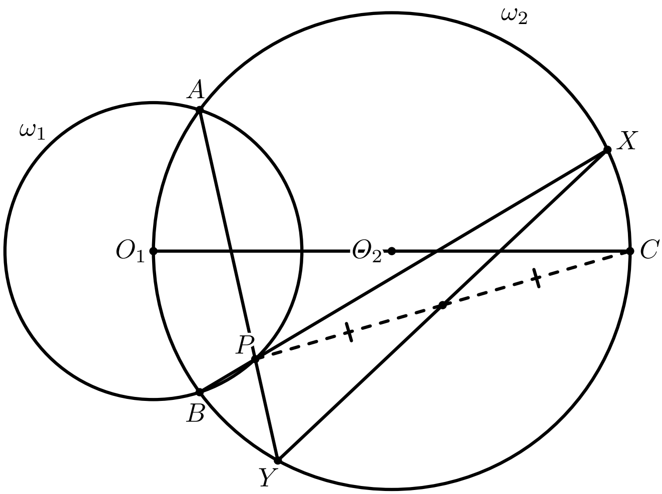
(→ p.21)

Solutions

1) Two circles ω_1 and ω_2 with centers O_1 and O_2 respectively intersect each other at points A and B, and point O_1 lies on ω_2. Let P be an arbitrary point lying on ω_1. Lines BP, AP and O_1O_2 cut ω_2 for the second time at points X, Y and C, respectively. Prove that quadrilateral $XPYC$ is a parallelogram.

Proposed by Iman Maghsoudi

Solution.



One can obtain

$$\angle APB = \frac{\widehat{AO_1B}}{2} + \frac{\widehat{XCY}}{2}. \tag{1}$$

Note that $\angle AO_2O_1 = \angle BO_2O_1$, therefore $\angle AO_1O_2 = \angle BO_1O_2$. Now since O_1 is the circumcenter of triangle APB, it is deduced that

$$\angle APB = 180^\circ - \frac{\angle AO_1B}{2} = 180^\circ - \frac{\overset{\frown}{AXC}}{2} = \frac{\overset{\frown}{ABC}}{2}. \tag{2}$$

Hence, according to equations 1 and 2 it is obtained that

$$\frac{\overset{\frown}{AO_1B}}{2} + \frac{\overset{\frown}{XCY}}{2} = \frac{\overset{\frown}{ABC}}{2} \implies \overset{\frown}{XCY} = \overset{\frown}{BYC},$$

thus $BX \parallel CY$, similarly $AY \parallel XC$. This imlplies that $XPYC$ is a parallelogram and the claim of the problem is proved. ■

2) Find all quadrilaterals $ABCD$ such that all four triangles DAB, CDA, BCD and ABC are similar to one-another.

Proposed by Morteza Saghafian

Answer. All rectangles.

Solution. First assume that $ABCD$ is a concave quadrilateral. Without loss of generality one can assume $\angle D > 180°$, in other words D lies inside of triangle ABC. Again without loss of generality one can assume that $\angle ABC$ is the maximum angle in triangle ABC. Therefore

$$\angle ADC = \angle ABC + \angle BAD + \angle BCD > \angle ABC.$$

Thus $\angle ADC$ is greater than all the angles of triangle ABC, so triangles ABC and ADC cannot be similar. So it is concluded that $ABCD$ must be convex.

Now let $ABCD$ be a convex quadrilateral. Without loss of generality one can assume that the $\angle B$ is the maximum angle in the quadrilateral. It can be written that

$$\angle ABC > \angle DBC, \quad \angle ABC \geq \angle ADC \geq \angle BCD.$$

Since triangles ABC and BCD are similar, it is implied that $\angle ABC = \angle BCD$ and similarly, all the angles of $ABCD$ are equal; Meaning $ABCD$ must be a rectangle. It is easy to see that indeed, all rectangles satisfy the conditions of the problem. ∎

3) Three circles ω_1, ω_2 and ω_3 pass through one common point, say P. The tangent line to ω_1 at P intersects ω_2 and ω_3 for the second time at points $P_{1,2}$ and $P_{1,3}$, respectively. Points $P_{2,1}$, $P_{2,3}$, $P_{3,1}$ and $P_{3,2}$ are similarly defined. Prove that the perpendicular bisector of segments $P_{1,2}P_{1,3}$, $P_{2,1}P_{2,3}$ and $P_{3,1}P_{3,2}$ are concurrent.

Proposed by Mahdi Etesamifard

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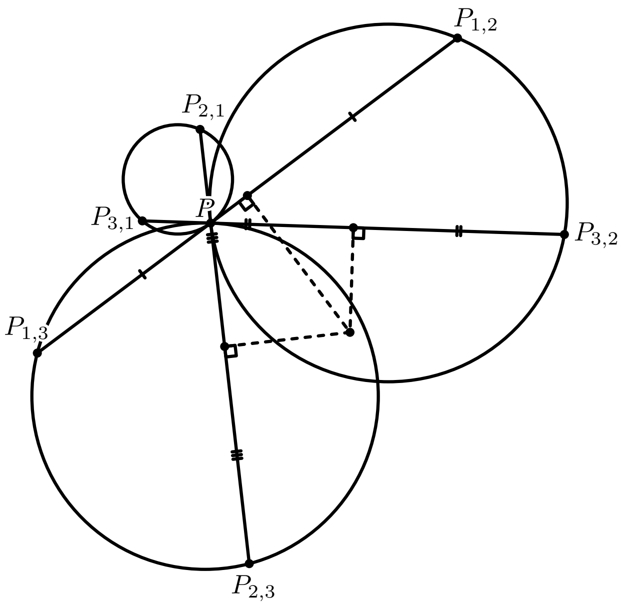
Solution.



First assume that no two of the lines $\ell_1 \equiv P_{2,1}P_{3,1}$, $\ell_2 \equiv P_{1,2}P_{3,2}$ and $\ell_3 \equiv P_{1,3}P_{2,3}$ are parallel; Consider triangle XYZ made by intersecting these lines, where

$$X \equiv \ell_2 \cap \ell_3,$$
$$Y \equiv \ell_1 \cap \ell_3,$$
$$Z \equiv \ell_2 \cap \ell_1.$$

Note that

$$\angle P_{3,2}P_{1,2}P = \angle P_{3,2}PP_{2,3} = \angle PP_{1,3}P_{2,3},$$

meaning $XP_{1,2} = XP_{1,3}$. Similarly, it is implied that $YP_{2,1} = YP_{2,3}$ and $ZP_{3,1} = ZP_{3,2}$. Therefore, the angle bisectors of angles YXZ, XYZ and

YZX are the same as the perpendicular bisectors of segments $P_{1,2}P_{1,3}$, $P_{2,1}P_{2,3}$ and $P_{3,1}P_{3,2}$; Thus, these three perpendicular bisectors are concurrent at the incenter of triangle XYZ, resulting in the claim of the problem.

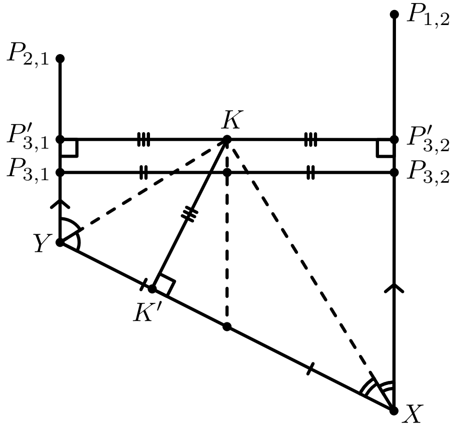
Now assume that at least two of the lines $\ell_1 = P_{2,1}P_{3,1}$, $\ell_2 = P_{1,2}P_{3,2}$ and $\ell_3 = P_{1,3}P_{2,3}$ are parallel; Without loss of generality assume that ℓ_1 and ℓ_2 are parallel. Similar to the previous case,

$$\angle P_{1,2}P_{3,2}P = \angle P_{1,2}PP_{2,1} = \angle P_{2,1}P_{3,1}P.$$

But since $\ell_1 \parallel \ell_2$, it is also true that

$$\angle P_{1,2}P_{3,2}P + \angle P_{2,1}P_{3,1}P = 180^\circ,$$

Hence $\angle P_{1,2}P_{3,2}P = \angle P_{2,1}P_{3,1}P = 90^\circ$. This equation immediately implies $\ell_3 \nparallel \ell_2$, because otherwise it would be deduced that $\ell_3 \perp P_{1,3}P_{1,2}$ and $\ell_2 \perp P_{1,3}P_{1,2}$, resulting in $P_{1,3}P_{1,2} \parallel P_{3,1}P_{3,2}$; Which is clearly not possible. Now consider trapezoid $XYP_{2,1}P_{1,2}$. The problem is now equivalent to show that the angle bisector of $\angle X$, angle bisector of $\angle Y$ and the perpendicular bisector of $P_{3,1}P_{3,2}$ concur. Note that ℓ_1 and ℓ_2 are parallel to the perpendicular bisector of $P_{3,1}P_{3,2}$, and in fact, the perpendicular bisector of $P_{3,1}P_{3,2}$ connects the midpoints of XY and $P_{3,1}P_{3,2}$. Now the claim of the problem is as simple as follows.



Claim. *In trapezoid $XYP_{2,1}P_{1,2}$, the angle bisector of $\angle X$, the angle bisector of $\angle Y$, and the mid-line of the trapezoid are concurrent.*

Proof. Let K be the intersection of the angle bisector of $\angle X$ and the angle bisector of $\angle Y$. Let $P'_{3,2}$, $P'_{3,1}$ and K' be the foot of perpendicular lines from K to lines $P_{1,2}X$, $P_{2,1}Y$ and XY, respectively. Since K lies on the angle bisector of $\angle X$, it is deduced that $KP'_{3,2} = KK'$, and similarly since K lies on the angle bisector of $\angle Y$, $KP'_{3,1} = KK'$; Thus $KP'_{3,1} = KP'_{3,2}$, meaning K lies on the mid-line of trapezoid $XYP_{2,1}P_{1,2}$. $\square$

This result leads to the conclusion of the problem. $\blacksquare$

4) Let $ABCD$ be a parallelogram and let K be a point on line AD such that $BK = AB$. Suppose that P is an arbitrary point on AB, and the perpendicular bisector of PC intersects the circumcircle of triangle APD at points X, Y. Prove that the circumcircle of triangle ABK passes through the orthocenter of triangle AXY.

Proposed by Iman Maghsoudi

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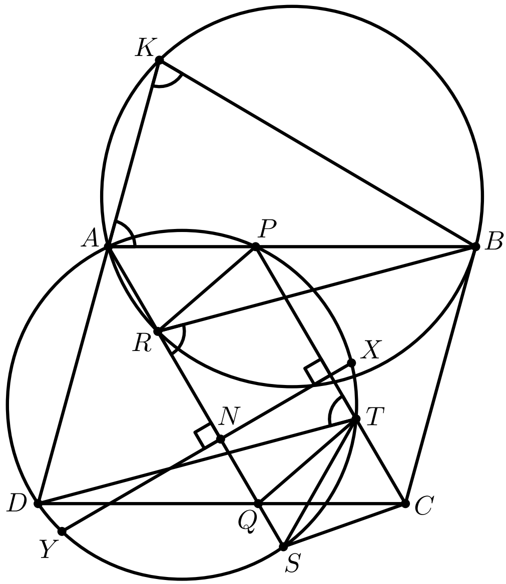
Solution. Let AN be the altitude of triangle AXY. Suppose that the circumcircle of triangle ABK intersects AN at the point R. It is enough to show that R is the orthocenter of triangle AXY.



Suppose that PC and AN intersects the circumcircle of triangle APD for the second time at point T and S, respectively and AN intersects CD at Q. We know that

$$\angle BRS = \angle AKB = \angle KAB = \angle PTD \Longrightarrow \angle BRS = \angle PTD \qquad (1)$$

Notice that XY is perpendicular to AN and PC, so $AN \parallel PC$. Also $AP \parallel CQ$ so $APCQ$ is parallelogram. Now we have

$$\left.\begin{array}{l} \angle BAR = \angle TCD \\ (1) \Longrightarrow \angle ARB = \angle CTD \\ AB = CD \end{array}\right\} \Longrightarrow \triangle ARB \cong \triangle CTD \Longrightarrow CT = AR.$$

So $PTQR$ is also a parallelogram. $APTS$ is an isosceles trapezoid so

$$CQ = AP = TS.$$

Therefore $TQSC$ is also an isosceles trapezoid. Finally

$$CS = TQ = PR \Longrightarrow CSRP \text{ is an isosceles trapezoid.}$$

Therefore, S and R are symmetric with respect to XY. So R should be the orthocenter of triangle AXY. ■

5) Let ABC be a triangle with $\angle A = 60°$. Points E and F are the foot of angle bisectors of vertices B and C respectively. Points P and Q are considered such that quadrilaterals $BFPE$ and $CEQF$ are parallelograms. Prove that $\angle PAQ > 150°$. (Consider the angle PAQ that does not contain side AB of the triangle.)

Proposed by Alireza Dadgarnia

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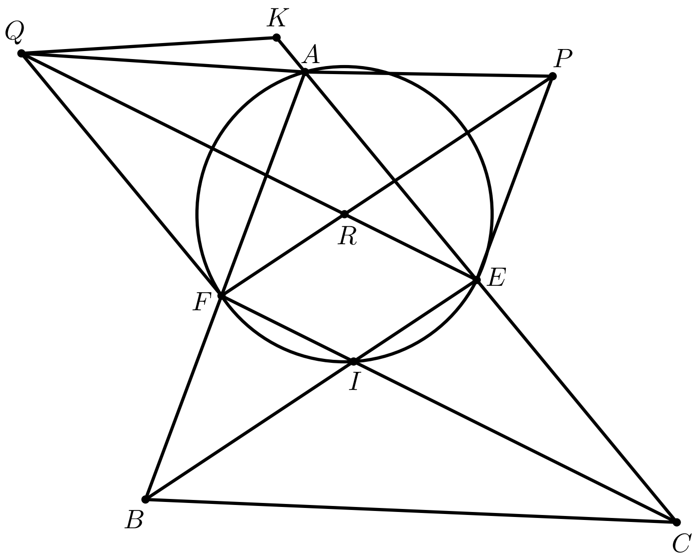
Solution. Let I and be the intersection point of lines BE and CF, and let R be the intersection point of lines QE and PF. It is easy to see that $\angle BIC = 120°$. Thus $AEIF$ is a cyclic quadrilateral and so

$$CE \cdot CA = CI \cdot CF \tag{1}$$

Also $\angle PRQ = \angle BIC = 120°$, therefore it suffices to show that at least on of the angles $\angle APR$ or $\angle AQR$ is greater than or equal to $30°$.



Assume the contrary, meaning both of these angles are less than $30°$. Hence there exists a point K on the extension of ray CA such that $\angle KQE = 30°$. Since $\angle IAC = 30°$ $\angle ACI = \angle KEQ$, it is deduced that $\triangle AIC \sim \triangle QKE$. This implies

$$\frac{CI}{CA} = \frac{KE}{QE} > \frac{AE}{CF} \implies AE < \frac{CF \cdot CI}{CA} \overset{(1)}{=} CE.$$

Similarly, it is obtained that $AF < BF$. On the other hand at least on of the angles $\angle ABC$ or $\angle ACB$ are not less than 60°. Without loss of generality one can assume that $\angle ABC \geq 60^\circ$ thus $AC \geq BC$ and according to angle bisector theorem it is obtained that $AF \geq BF$, which is a contradiction. Hence the claim of the problem. $\blacksquare$

Advanced Level

Problems

1) Circles ω_1 and ω_2 intersect each other at points A and B. Point C lies on the tangent line from A to ω_1 such that $\angle ABC = 90°$. Arbitrary line ℓ passes through C and cuts ω_2 at points P and Q. Lines AP and AQ cut ω_1 for the second time at points X and Z respectively. Let Y be the foot of altitude from A to ℓ. Prove that points X, Y and Z are collinear.

(→ p.26)

2) Is it true that in any convex n-gon with $n > 3$, there exists a vertex and a diagonal passing through this vertex such that the angles of this diagonal with both sides adjacent to this vertex are acute?

(→ p.28)

3) Circles ω_1 and ω_2 have centres O_1 and O_2, respectively. These two circles intersect at points X and Y. AB is common tangent line of these two circles such that A lies on ω_1 and B lies on ω_2. Let tangents to ω_1 and ω_2 at X intersect O_1O_2 at points K and L, respectively. Suppose that line BL intersects ω_2 for the second time at M and line AK intersects ω_1 for the second time at N. Prove that lines AM, BN and O_1O_2 concur.

(→ p.29)

4) Given an acute non-isosceles triangle ABC with circumcircle Γ. M is the midpoint of segment BC and N is the midpoint of $\widehat{BC}$ of Γ (the one that doesn't contain A). X and Y are points on Γ such that $BX \parallel CY \parallel AM$. Assume there exists point Z on segment BC such that circumcircle of triangle XYZ is tangent to BC. Let ω be the circumcircle of triangle ZMN. Line AM meets ω for the second time at P. Let K be a point on ω such that $KN \parallel AM$, ω_b be a circle that passes through B, X and tangents to BC and ω_c be a circle that passes through C, Y and tangents to BC. Prove that circle with center K and radius KP is tangent to 3 circles ω_b, ω_c and Γ.

(→ p.30)

5) Let points A, B and C lie on the parabola Δ such that the point H, orthocenter of triangle ABC, coincides with the focus of parabola Δ. Prove that by changing the position of points A, B and C on Δ so that the orthocenter remain at H, inradius of triangle ABC remains unchanged.

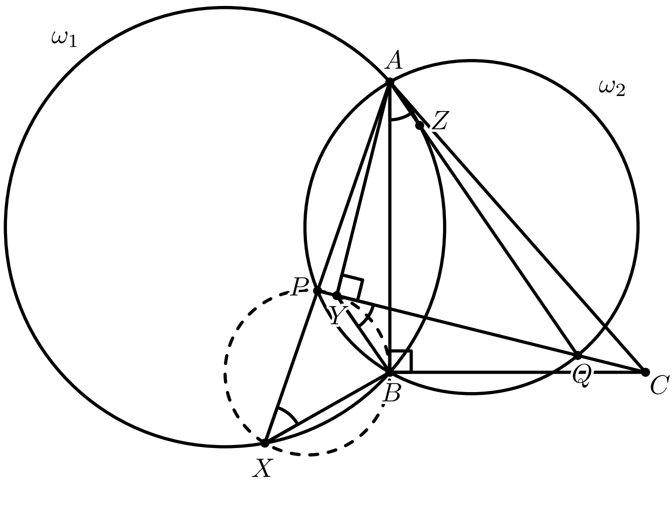
(→ p.33)

Solutions

1) Circles ω_1 and ω_2 intersect each other at points A and B. Point C lies on the tangent line from A to ω_1 such that $\angle ABC = 90°$. Arbitrary line ℓ passes through C and cuts ω_2 at points P and Q. Lines AP and AQ cut ω_1 for the second time at points X and Z respectively. Let Y be the foot of altitude from A to ℓ. Prove that points X, Y and Z are collinear.

Proposed by Iman Maghsoudi

Solution.



Since $\angle AYC = \angle ABC = 90°$, it is concluded that $AYBC$ is a cyclic quadrilateral. Hence

$$\angle BYC = \angle BAC = \angle BXA = \angle BXP.$$

So $PYBX$, and similarly $QBYZ$ are cyclic quadrilaterals. Therefore it is implied that

$$\angle BYX = \angle BPX = \angle AQB = \angle ZQB = 180^\circ - \angle ZYB.$$

Meaning points X, Y and Z are collinear. $\blacksquare$

2) Is it true that in any convex n-gon with $n > 3$, there exists a vertex and a diagonal passing through this vertex such that the angles of this diagonal with both sides adjacent to this vertex are acute?

Proposed by Boris Frenkin - Russia

Answer. Yes, it is true.

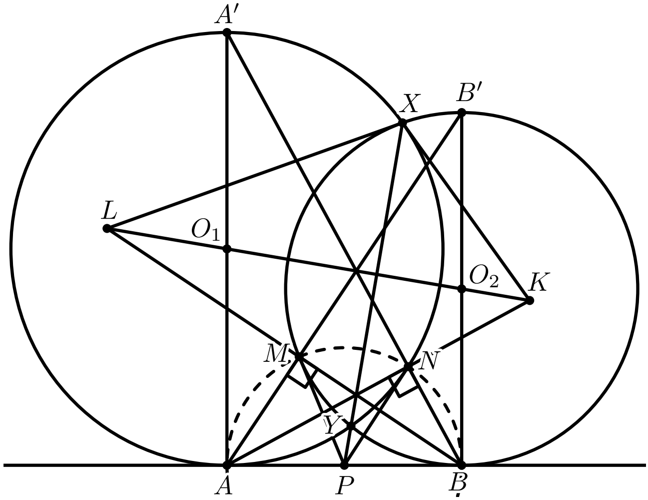
Solution. Suppose the answer is no. Given a convex n-gon ($n > 3$), consider its longest diagonal AD (if the longest diagonal is not unique, choose an arbitrary on among them). Let B and C be the vertices neighboring to A. Without loss of generality assume that $\angle BAD \geq 90°$. This means $BD > AD$, so BD is not a diagonal and hence is a side of the n-gon. Furthermore, $\angle ADB < 90°$. Let C' be the vertex neighboring to D and distinct from B. Then $\angle ADC' \geq 90°$. Similarly, $AC' > AD$, so AC' is a side, $C' \equiv C$ and $n = 4$. Angles BAC and BDC are obtuse, so BC is longer than AC and BD, hence $BC > AD$ and AD is not the longest diagonal, a contradiction. Hence the claim. ■

3) Circles ω_1 and ω_2 have centres O_1 and O_2, respectively. These two circles intersect at points X and Y. AB is common tangent line of these two circles such that A lies on ω_1 and B lies on ω_2. Let tangents to ω_1 and ω_2 at X intersect O_1O_2 at points K and L, respectively. Suppose that line BL intersects ω_2 for the second time at M and line AK intersects ω_1 for the second time at N. Prove that lines AM, BN and O_1O_2 concur.

Proposed by Dominik Burek - Poland

Solution.

Let P be the midpoint of AB; Since P has the same power with respect to both circles, it lies on the radical axis of them, which is line XY.

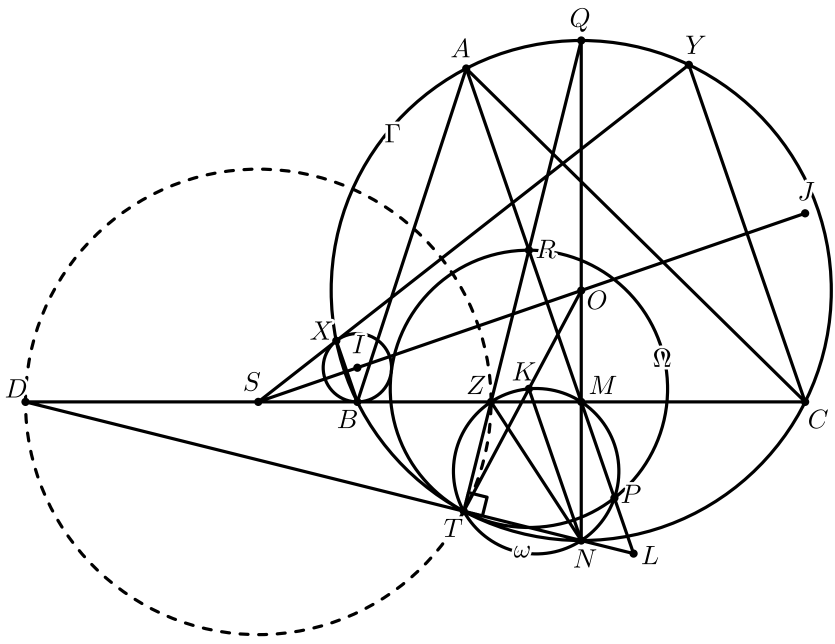


According to the symmetry, KY is tangent to ω_1, therefore XY is the polar of K with respect to ω_1. Since P lies on XY, the polar of P passes through K, and similarly, it also passes through A; Meaning AK is the polar of P with respect to ω_1 and PN is tangent to ω_1. Similarly, PM is tangent to ω_2; Thus points A, B, M and N lie on a circle with center P and $\angle AMB = \angle ANP = 90°$. Let A' be the antipode of A in circle ω_1, and let B' be the antipode of B. Line BN passes through A' and line AM passes through B'. Note that $AA'B'B$ is a trapezoid and O_1 and O_2 are the midpoints of its bases; Hence $A'B$, $B'A$ and O_1O_2 are concurrent, resulting in the claim of the problem. ■

4) Given an acute non-isosceles triangle ABC with circumcircle Γ. M is the midpoint of segment BC and N is the midpoint of $\widehat{BC}$ of Γ (the one that doesn't contain A). X and Y are points on Γ such that $BX \parallel CY \parallel AM$. Assume there exists point Z on segment BC such that circumcircle of triangle XYZ is tangent to BC. Let ω be the circumcircle of triangle ZMN. Line AM meets ω for the second time at P. Let K be a point on ω such that $KN \parallel AM$, ω_b be a circle that passes through B, X and tangents to BC and ω_c be a circle that passes through C, Y and tangents to BC. Prove that circle with center K and radius KP is tangent to 3 circles ω_b, ω_c and Γ.

Proposed by Tran Quan - Vietnam

Solution. Let I, J and O be the centers of circles ω_b, ω_c and Γ, respectively. It's easy to see that I, J and O are collinear. Let S be the intersection of IJ and BC. Since X and Y are symmetrical to B and C with respect to IJ, it is implied that S, X and Y are collinear. Let T be a point on $\widehat{BC}$ of Γ (the one that doesn't contain A) such that ST is tangent to Γ. TZ meets Γ again at Q.



Since

$$ST^2 = SX \cdot SY = SZ^2,$$

ZT is the interior angle bisector of $\angle BTC$, it is concluded that Q is the midpoint of $\widehat{BAC}$ of Γ. This leads to $ZT \perp NT$, resulted in T lies on ω. Let R be the intersection of AM and TQ, and Ω be the circumcircle of triangle RTP. Since

$$\angle RPT = \angle MPT = \angle SZT = \angle STR,$$

ST is tangent to Ω. Therefore Ω is tangent to Γ at T. We will prove that K is the center of Ω. Now

$$\begin{cases} SOMT \text{ is cyclic } \implies \angle OSM = \angle OTM \\ OS \perp AM \text{ and } MS \perp OM \implies \angle OMA = \angle OSM = \angle OTM \\ \angle OTQ = \angle OQT \end{cases}$$

Which implies

$$\angle MTR = \angle OTQ + \angle OTM = \angle OQT + \angle OMA = \angle MRT,$$

and thus $MR = MT$. Let L be the intersection of AM and TN. Since $\triangle RTL$ is a right triangle at T, it is concluded that M is the midpoint of RL.

$$\begin{aligned} KN \parallel ML &\implies \angle MTN = \angle MLT = \angle KNT \\ &\implies KM \parallel TN \\ &\implies KM \perp RT \end{aligned}$$

Thus, MK is the perpendicular bisector of RT. Note that

$$\angle ZPR = \angle ZTM = \angle ZRP \implies ZR = ZP.$$

Since ZN is the diameter of ω it is implied that

$$ZK \perp KN \implies ZK \perp RP.$$

Therefore ZK is the perpendicular bisector of RP. Hence K is the center of Ω.

We will prove Ω is tangent to ω_b and ω_c. Let D be the intersection of TN and BC, and denote (S, SZ) to be circle with center S and radius SZ. Since TZ is the interior angle bisector of $\angle BTC$, we have TD is the exterior angle bisector of the same angle. This leads to $(DZ, BC) = -1$. Since M is the midpoint of BC, we have

$$MB^2 = MZ \cdot MD,$$

which implies M lies on the radical axis of ω_b and (S, SZ). Combining with $MA \perp SI$, we have MA is the radical axis of ω_b and (S, SZ). Thus, the powers of point R with respect to ω_b and (S, SZ) are equal. Invert about the circle centered at R with radius

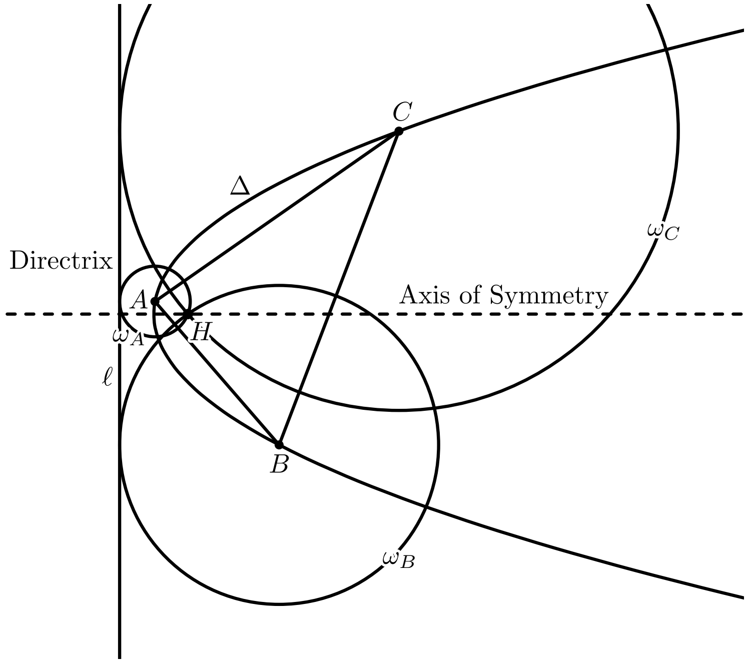
$$r = \sqrt{RZ \cdot RT},$$

which inverts $\Omega \mapsto ZM \equiv BC$ and $\omega_b \mapsto \omega_b$. Since circle ω_b is tangent to BC, we have Ω is tangent to ω_b.
Analogously, we have Ω is tangent to ω_c. This completes the proof. $\blacksquare$

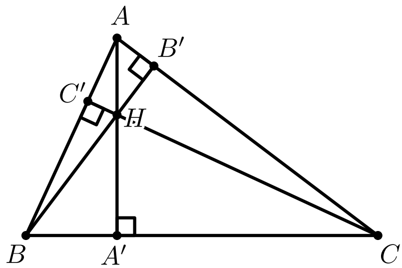
5) Let points A, B and C lie on the parabola Δ such that the point H, orthocenter of triangle ABC, coincides with the focus of parabola Δ. Prove that by changing the position of points A, B and C on Δ so that the orthocenter remain at H, inradius of triangle ABC remains unchanged.

Proposed by Mahdi Etesamifard

Solution. Since H coincides with the focus of parabola Δ, the circles $w_A = (A, AH)$, $w_B = (B, BH)$ and $w_C = (C, CH)$ are tangent to line ℓ, the directrix of Δ.



Now consider triangle ABC.



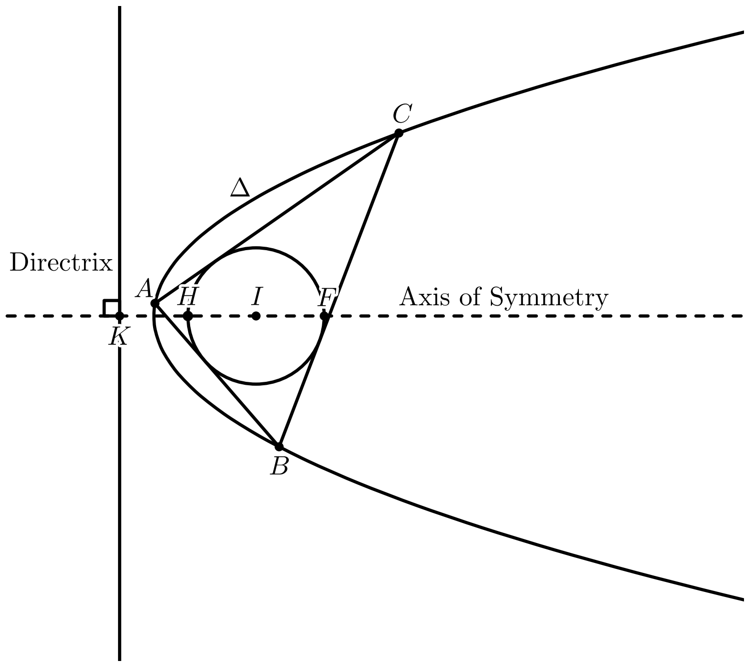
It is well-known that

$$HA \cdot HA' = HB \cdot HB' = HC \cdot HC' = t.$$

Also

$$\left.\begin{array}{l} HA = 2R\cos A \\ HA' = 2R\cos B\cos C \end{array}\right\} \implies t = 4R^2\cos A\cos B\cos C. \tag{1}$$

Inversion with center H and inversion radius $-2t$, inverts the three circles w_A, w_B and w_C to lines BC, AC and AB respectively. In this inversion, line ℓ inverts to incircle of triangle ABC. Therefore $IH \perp \ell$, thus point I lies on axis of symmetry of Δ. Also point H lies on the incircle of triangle ABC. Hence $HI = r$.



As a result, if orthocenter of ABC lies on its incircle; Also

$$\begin{aligned} & HI^2 = 2r^2 - 4R^2\cos A\cos B\cos C \\ \implies & r^2 = 2r^2 - 4R^2\cos A\cos B\cos C \\ \implies & r^2 = 4R^2\cos A\cos B\cos C \end{aligned}$$

According to (1), it is implied that $r^2 = t = HA \cdot HA'$. In inversion, points K and F are invert points, thus

$$\overrightarrow{HK} \cdot \overrightarrow{HF} = -2t = -2r^2 \implies HK = r.$$

Which gives the result that inradius of triangle ABC is constant. $\blacksquare$