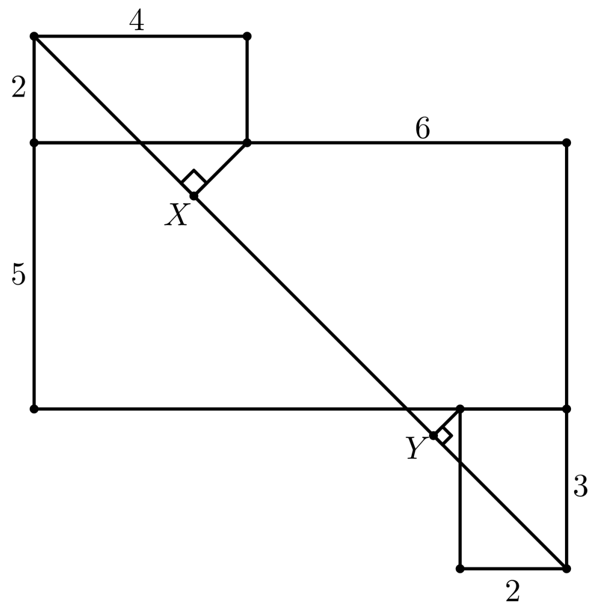
The problems of 5^{th} IGO along with their solutions

Intermediate Level

Problems:

1. There are three rectangles in the following figure. The lengths of some segments are shown. Find the length of the segment XY.



2. In convex quadrilateral $ABCD$, the diagonals AC and BD meet at the point P. We know that $\angle DAC = 90°$ and $2\angle ADB = \angle ACB$. If we have $\angle DBC + 2\angle ADC = 180°$ prove that $2AP = BP$.

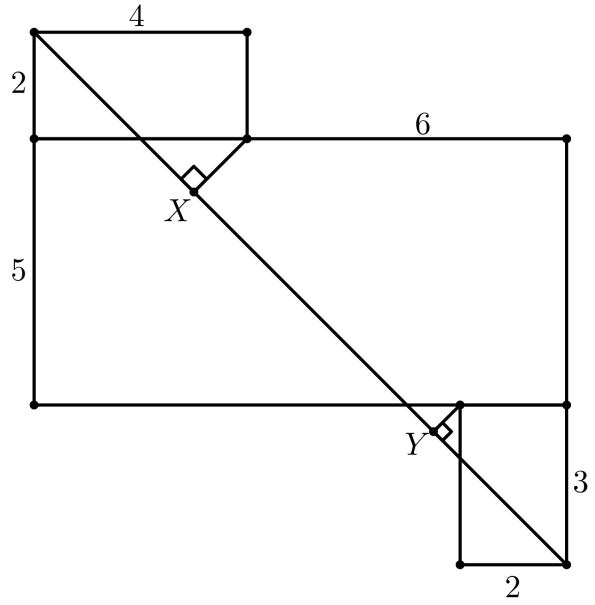
3. Let ω_1, ω_2 be two circles with centers O_1 and O_2, respectively. These two circles intersect each other at points A and B. Line O_1B intersects ω_2 for the second time at point C, and line O_2A intersects ω_1 for the second time at point D . Let X be the second intersection of AC and ω_1. Also Y is the second intersection point of BD and ω_2. Prove that $CX = DY$.

4. We have a polyhedron all faces of which are triangle. Let P be an arbitrary point on one of the edges of this polyhedron such that P is not the midpoint or endpoint of this edge. Assume that $P_0 = P$. In each step, connect P_i to the centroid of one of the faces containing it. This line meets the perimeter of this face again at point P_{i+1}. Continue this process with P_{i+1} and the other face containing P_{i+1}. Prove that by continuing this process, we cannot pass through all the faces. (The centroid of a triangle is the point of intersection of its medians.)

5. Suppose that $ABCD$ is a parallelogram such that $\angle DAC = 90°$. Let H be the foot of perpendicular from A to DC, also let P be a point along the line AC such that the line PD is tangent to the circumcircle of the triangle ABD. Prove that $\angle PBA = \angle DBH$.

Solutions:

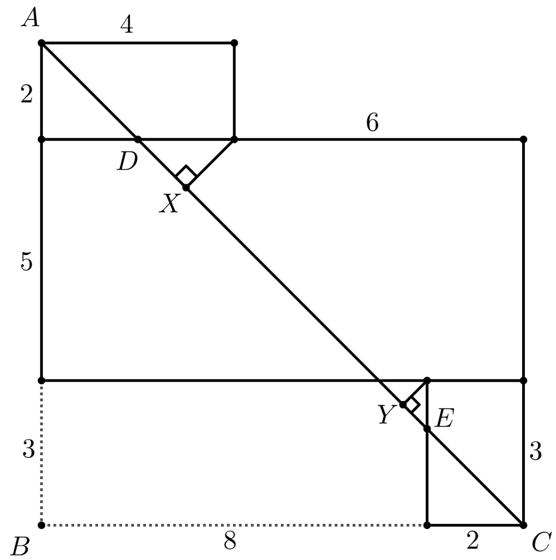
1. There are three rectangles in the following figure. The lengths of some segments are shown. Find the length of the segment XY.



Proposed by Hirad Aalipanah

Let us continue the rectangular sides to get the ABC triangle. Because $AB = BC$ we can say that $\angle BCA = \angle BAC = 45°$. Therefore, we can determine some of the segments using the Pythagoras's theorem such as $AD = 2\sqrt{2}$, $DX = \sqrt{2}$, $CE = 2\sqrt{2}$ and $EY = \frac{\sqrt{2}}{2}$. So, we have

$$XY = AC - AD - DX - CE - EY = 10\sqrt{2} - 2\sqrt{2} - \sqrt{2} - 2\sqrt{2} - \frac{\sqrt{2}}{2} = \frac{9\sqrt{2}}{2}$$



2. In convex quadrilateral $ABCD$, the diagonals AC and BD meet at the point P. We know that $\angle DAC = 90^\circ$ and $2\angle ADB = \angle ACB$. If we have $\angle DBC + 2\angle ADC = 180^\circ$ prove that $2AP = BP$.

Proposed by Iman Maghsoudi

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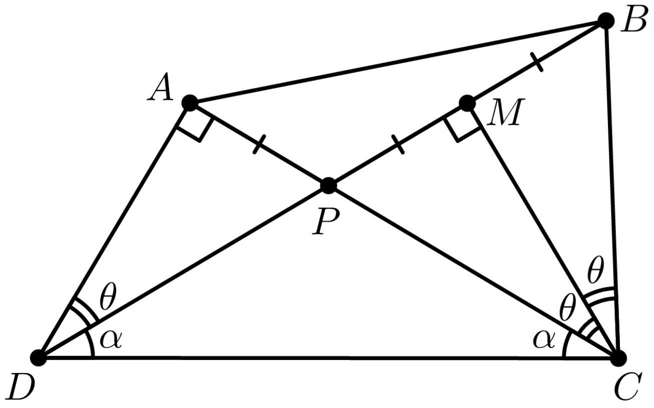
Solution.



Let M be the intersection point of the angle bisector of $\angle PCB$ with segment PB. Since $\angle PCM = \angle PDA = \theta$ and $\angle APD = \angle MPC$, we get that $\triangle PMC \sim \triangle PAD$, which means $\angle PMC = 90^\circ$.
Now in triangle CPB, the angle bisector of vertex C is the same as the altitude from C, this means CPB is an isosceles triangle and so $PM = MB, PC = CB$.
In triangle DBC, we have

$$\widehat{DBC} + 2\theta + \widehat{PCD} + \widehat{PDC} = 180^\circ.$$

This along with the assumption that $\angle DBC + 2\angle ADC = 180^\circ$, implies $\angle PCD = \angle PDC$. Therefore $PC = PD$ and so $\triangle PMC \cong \triangle PAD$, hence $AP = PM = \frac{PB}{2}$. ■

3. Let ω_1, ω_2 be two circles with centers O_1 and O_2, respectively. These two circles intersect each other at points A and B. Line O_1B intersects ω_2 for the second time at point C, and line O_2A intersects ω_1 for the second time at point D . Let X be the second intersection of AC and ω_1. Also Y is the second intersection point of BD and ω_2. Prove that $CX = DY$.

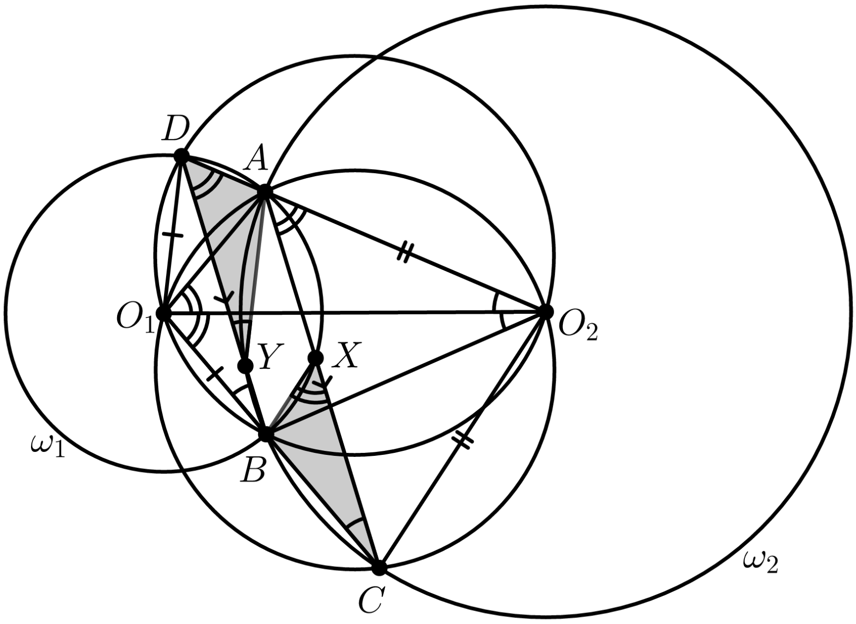
Proposed by Alireza Dadgarnia

Solution. First, we use a well-known lemma.

Lemma. *Let $PQRS$ be a convex quadrilateral with $RQ = RS$, $\angle RPQ = \angle RPS$ and $PQ \neq PS$. Then $PQRS$ is cyclic.*

Proof. Assume the contrary, and let $P' \neq P$ be the intersection point of the circle passing through R, S, Q with line PR.
Since $P'QRS$ is cyclic and $RQ = RS$, we get $\angle SP'R = \angle QP'R$. Now let's considerate on triangles $SP'P$ and $QP'P$. In these two triangles we have $\angle SP'P = \angle QP'P$ and also $\angle P'PQ = \angle P'PS$. This means these two triangles are equal, hence $PQ = PS$, which is a contradiction. So the lemma is proved.

Back to the problem.



Triangles ADY and BXC are similar, because

$$\widehat{ADY} = \widehat{BXC} = 180^\circ - \widehat{BXA},$$

and

$$\widehat{DYA} = \widehat{BCX} = 180^\circ - \widehat{AYB}.$$

Note that O_2 lies on the angle bisector of $\angle AO_1B$, $O_2A = O_2C$ and also $O_1A \neq O_1C$. So we can use the lemma and conclude that O_1AO_2C is cyclic. Similarly, we get that O_2BO_1D is cyclic.

$$\widehat{AYD} = 180^\circ - \widehat{AYB} = \widehat{O_1CA} = \widehat{O_1O_2A} = \widehat{O_1BD}.$$

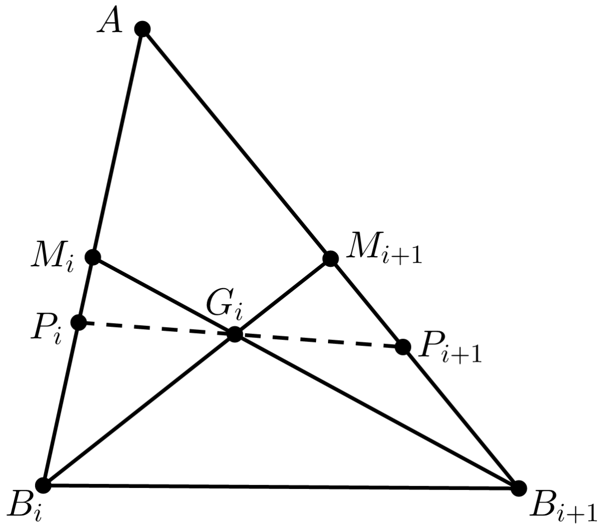
Which means $AC \parallel BD$ and so $AY = BC$. But since $\triangle ADY \sim \triangle BXC$, we get that these two triangle are equal and so $CX = DY$. ■

4. We have a polyhedron all faces of which are triangle. Let P be an arbitrary point on one of the edges of this polyhedron such that P is not the midpoint or endpoint of this edge. Assume that $P_0 = P$. In each step, connect P_i to the centroid of one of the faces containing it. This line meets the perimeter of this face again at point P_{i+1}. Continue this process with P_{i+1} and the other face containing P_{i+1}. Prove that by continuing this process, we cannot pass through all the faces. (The centroid of a triangle is the point of intersection of its medians.)

Proposed by Mahdi Etesamifard - Morteza Saghafian

Solution. Suppose that AB is the edge that P lies on. Let M be the midpoint of AB and without loss of generality, assume that P lies between B and M. We will prove that it is impossible to pass through a face which doesn't contain A. (Such face exists in any polyhedron)

Let $B = B_0, B_1, B_2, \ldots$ be the vertices adjacent to A in this order. Let M_i be the midpoint of AB_i. By using induction, we prove that for each i, P_i lies on edge AB_i, between B_i and M_i. For $i = 0$ the claim is true. Now assume the claim for i and consider the triangle AB_iB_{i+1} with centroid G_i.



Since P_i lies between M_i and B_i, we get that P_iG_i lies between M_iG_i and B_iG_i, which are the medians of this triangle. So P_{i+1} lies on AB_{i+1}, between M_{i+1} and B_{i+1}. So the claim is proved.

We proved that P_i's lie on AB_i's, so the sequence of points P_i goes around A and therefore does not pass through a face which doesn't contain A. ■

5. Suppose that $ABCD$ is a parallelogram such that $\angle DAC = 90°$. Let H be the foot of perpendicular from A to DC, also let P be a point along the line AC such that the line PD is tangent to the circumcircle of the triangle ABD. Prove that $\angle PBA = \angle DBH$.

Proposed by Iman Maghsoudi

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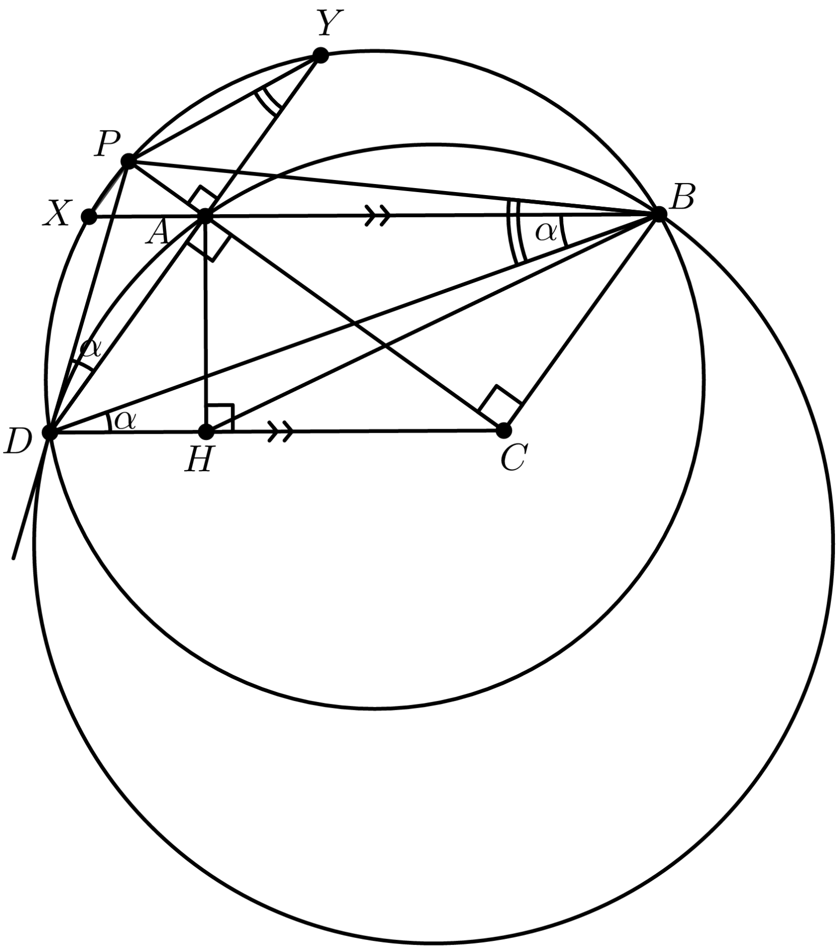

Suppose that AB, AD meet the circumcircle of triangle PDB for the second time at points X, Y respectively. Let $\angle CDB = \alpha$ and $\angle ADB = \theta$. Therefore, we have $\angle ABD = \alpha$, and so $\angle ADP = \alpha$.

Also $\angle PDB = \angle PXB = \alpha + \theta$, and $\angle PAX = \angle ACD = \angle DAH$. Which implies

$$\overset{\triangle}{APX} \sim \overset{\triangle}{ADH} \Longrightarrow \frac{AP}{AH} = \frac{AX}{AD},$$

$$\overset{\triangle}{XAD} \sim \overset{\triangle}{YAB} \Longrightarrow \frac{AY}{AB} = \frac{AX}{AD},$$

$$\Longrightarrow \frac{AP}{AH} = \frac{AY}{AB}.$$

Now since $\angle HAB = \angle PAY = 90°$, It can be written that $\overset{\triangle}{APY} \sim \overset{\triangle}{AHB}$.

$$\Longrightarrow \widehat{HBA} = \widehat{PYA} = \widehat{PBD} \Longrightarrow \widehat{PBA} = \widehat{DBH}.$$

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