4th Iranian Geometry Olympiad

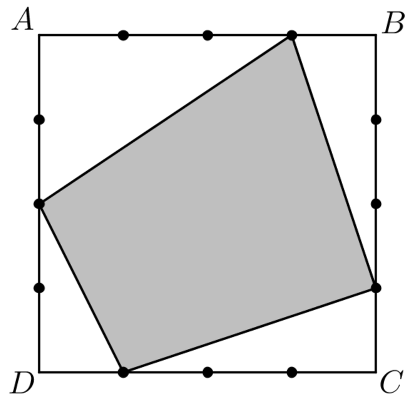
Problems and Solutions - 2017

This booklet is prepared by Hirad Aalipanah, Iman Maghsoudi.
With special thanks to Morteza Saghafian, Mahdi Etesami Fard,
Davood Vakili, Hooman Fattahimoghaddam.

Ministry of education, Islamic Republic of Iran.
www.ysc.ac.ir - www.igo-official.ir

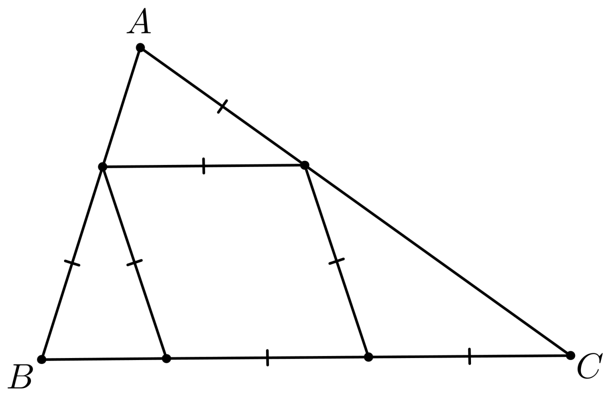
Problems of 4th Iranian Geometry Olympiad 2017 (Elementary)

1. Each side of square $ABCD$ with side length of 4 is divided into equal parts by three points. Choose one of the three points from each side, and connect the points consecutively to obtain a quadrilateral. Which numbers can be the area of this quadrilateral? Just write the numbers without proof.



Proposed by Hirad Aalipanah

2. Find the angles of triangle ABC.



Proposed by Morteza Saghafian

3. In the regular pentagon $ABCDE$, the perpendicular at C to CD meets AB at F. Prove that $AE + AF = BE$.

Proposed by Alireza Cheraghi

4. $P_1, P_2,..., P_{100}$ are 100 points on the plane, no three of them are collinear. For each three points, call their triangle clockwise if the increasing order of them is in clockwise order. Can the number of clockwise triangles be exactly 2017?

Proposed by Morteza Saghafian

5. In the isosceles triangle ABC ($AB = AC$), let l be a line parallel to BC through A. Let D be an arbitrary point on l. Let E, F be the feet of perpendiculars through A to BD, CD respectively. Suppose that P, Q are the images of E, F on l. Prove that $AP + AQ \leq AB$.

Proposed by Morteza Saghafian

Problems of 4th Iranian Geometry Olympiad 2017 (Intermediate)

1. Let ABC be an acute-angled triangle with $A = 60°$. Let E, F be the feet of altitudes through B, C respectively. Prove that $CE - BF = \frac{3}{2}(AC - AB)$.

Proposed by Fatemeh Sajadi

2. Two circles ω_1, ω_2 intersect at A, B. An arbitrary line through B meets ω_1, ω_2 at C, D respectively. The points E, F are chosen on ω_1, ω_2 respectively so that $CE = CB$, $BD = DF$. Suppose that BF meets ω_1 at P, and BE meets ω_2 at Q. Prove that A, P, Q are collinear.

Proposed by Iman Maghsoudi

3. On the plane, n points are given ($n > 2$). No three of them are collinear. Through each two of them the line is drawn, and among the other given points, the one nearest to this line is marked (in each case this point occurred to be unique). What is the maximal possible number of marked points for each given n?

Proposed by Boris Frenkin (Russia)

4. In the isosceles triangle ABC ($AB = AC$), let l be a line parallel to BC through A. Let D be an arbitrary point on l. Let E, F be the feet of perpendiculars through A to BD, CD respectively. Suppose that P, Q are the images of E, F on l. Prove that $AP + AQ \leq AB$.

Proposed by Morteza Saghafian

5. Let X, Y be two points on the side BC of triangle ABC such that $2XY = BC$. (X is between B, Y) Let AA' be the diameter of the circumcircle of triangle AXY. Let P be the point where AX meets the perpendicular from B to BC, and Q be the point where AY meets the perpendicular from C to BC. Prove that the tangent line from A' to the circumcircle of AXY passes through the circumcenter of triangle APQ.

Proposed by Iman Maghsoudi

Problems of 4th Iranian Geometry Olympiad 2017 (Advanced)

1. In triangle ABC, the incircle, with center I, touches the side BC at point D. Line DI meets AC at X. The tangent line from X to the incircle (different from AC) intersects AB at Y. If YI and BC intersect at point Z, prove that $AB = BZ$.

Proposed by Hooman Fattahimoghaddam

2. We have six pairwise non-intersecting circles that the radius of each is at least one. Prove that the radius of any circle intersecting all the six circles, is at least one.

Proposed by Mohammad Ali Abam– Morteza Saghafian

3. Let O be the circumcenter of triangle ABC. Line CO intersects the altitude through A at point K. Let P, M be the midpoints of AK, AC respectively. If PO intersects BC at Y, and the circumcircle of triangle BCM meets AB at X, prove that $BXOY$ is cyclic.

Proposed by Ali Daeinabi – Hamid Pardazi

4. Three circles ω_1, ω_2, ω_3 are tangent to line l at points A, B, C (B lies between A, C) and ω_2 is externally tangent to the other two. Let X, Y be the intersection points of ω_2 with the other common external tangent of ω_1, ω_3. The perpendicular line through B to l meets ω_2 again at Z. Prove that the circle with diameter AC touches ZX, ZY.

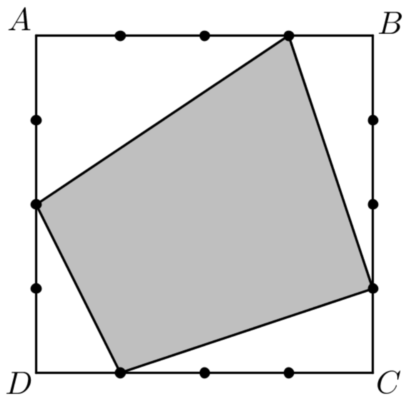
Proposed by Iman Maghsoudi – Siamak Ahmadpour

5. Sphere S touches a plane. Let A, B, C, D be four points on this plane such that no three of them are collinear. Consider the point A' such that S is tangent to the faces of tetrahedron $A'BCD$. Points B', C', D' are defined similarly. Prove that A', B', C', D' are coplanar and the plane $A'B'C'D'$ touches S.

Proposed by Alexey Zaslavsky (Russia)

Solutions of 4th Iranian Geometry Olympiad 2017 (Elementary)

1. Each side of square $ABCD$ with side length of 4 is divided into equal parts by three points. Choose one of the three points from each side, and connect the points consecutively to obtain a quadrilateral. Which numbers can be the area of this quadrilateral? Just write the numbers without proof.



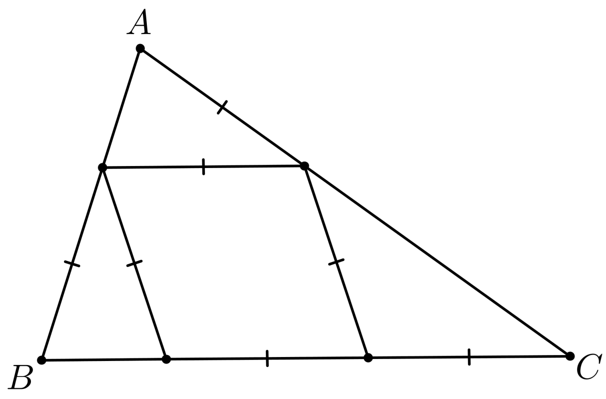
Proposed by Hirad Aalipanah

Solution.

To find the area of the quadrilaterals, it's enough to find sum of the areas of four right triangles and subtract it from area of the square. Finally, these numberes will be found:

$$6,\ 7,\ 7.5,\ 8,\ 8.5,\ 9,\ 10$$

2. Find the angles of triangle ABC.

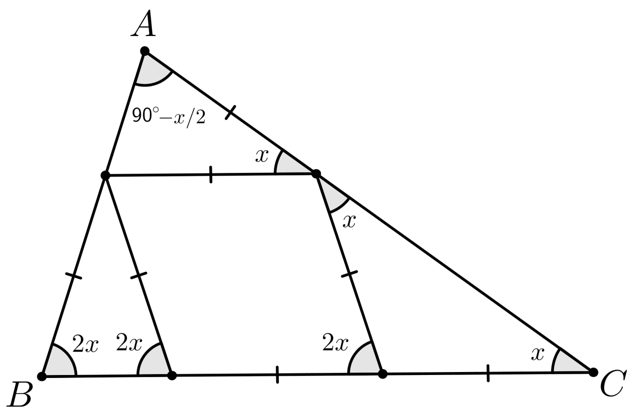


Proposed by Morteza Saghafian

— —

Solution.

Let $\angle ACB = x$. The quadrilateral with equal sides in triangle ABC is a rhombus so it has parallel sides.



According to the angles shown in the figure, we can say that sum of the angles of triangle ABC is equal to $180°$.

$$(90° - \frac{x}{2}) + 2x + x = 180° \quad \Rightarrow \quad x = 36°$$

$$\Rightarrow \angle A = 72° \quad , \quad \angle B = 72° \quad , \quad \angle C = 36°$$

3. In the regular pentagon $ABCDE$, the perpendicular at C to CD meets AB at F. Prove that $AE + AF = BE$.

Proposed by Alireza Cheraghi

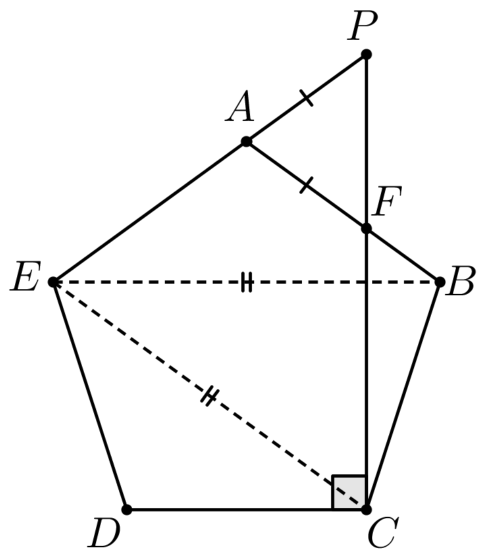
Solution.

Suppose that P is the intersection of AE and FC. We know that:

$$\angle ECD = 36^\circ \quad \Rightarrow \quad \angle ECP = 54^\circ \;,\; \angle AEC = 72^\circ \quad \Rightarrow \quad \angle EPC = 54^\circ$$

Therefore, $CE = PE$. On the other hand, we know that $\angle ECB = \angle EBC = 72^\circ$. Therefore, $BE = CE = PE$. Also we have $\angle EAB = 108^\circ$ so $\angle AFP = \angle APF = 54^\circ$. It means that $AF = AP$.

$$\Rightarrow \quad AE + AF = AE + AP = PE = CE = BE \;\blacksquare$$



4. $P_1, P_2, ..., P_{100}$ are 100 points on the plane, no three of them are collinear. For each three points, call their triangle clockwise if the increasing order of them is in clockwise order. Can the number of clockwise triangles be exactly 2017?

Proposed by Morteza Saghafian

Solution.
First, suppose that $P_1, P_2, ..., P_{100}$ are located on a circle in a counter-clockwise order. In this case, the number of clockwise triangles is zero. Now, start to move the points, when a point P_i passes through a line P_jP_k, the state of triangle $P_iP_jP_k$ (clockwise or counter-clockwise) changes and the state of the other triangles does not change (assume that the points move in such a way that no point passes through two lines at the same time). So the number of clockwise triangles changes by one. Suppose that the points $P_1, P_2, ..., P_{100}$ move to obtain their clockwise order on a circle. In this case, the number of clockwise triangles is $\binom{100}{3}$ which is greater than 2017. In this process, the number of clockwise triangles changes one by one. So there is a moment in which the number of clockwise triangles is exactly 2017. ■

5. In the isosceles triangle ABC ($AB = AC$), let l be a line parallel to BC through A. Let D be an arbitrary point on l. Let E, F be the feet of perpendiculars through A to BD, CD respectively. Suppose that P, Q are the images of E, F on l. Prove that $AP + AQ \leq AB$.

Proposed by Morteza Saghafian

Solution1.

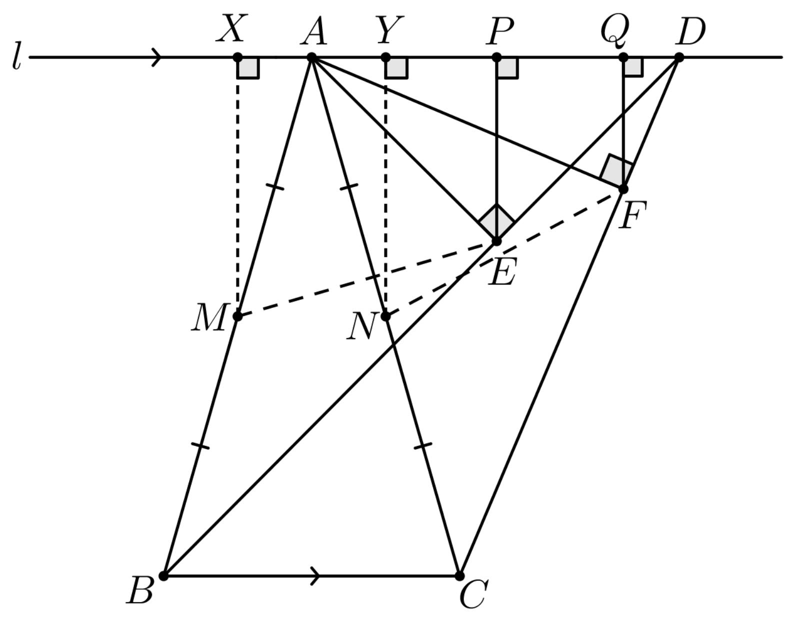
Suppose that M, N are the midpoints of AB , AC respectively. We have:

$$\angle AEB = \angle AFC = 90^\circ \quad \Rightarrow \quad ME = NF = \frac{AB}{2} = \frac{AC}{2}$$

Let X, Y be the feet of perpendiculars through M, N to line l respectively. We know that two triangles AMA and ANY are equal so $AX = AY$.

$$XP = AX + AP \leq ME = \frac{AB}{2} \quad , \quad AQ - AY = YQ \leq NF = \frac{AC}{2}$$

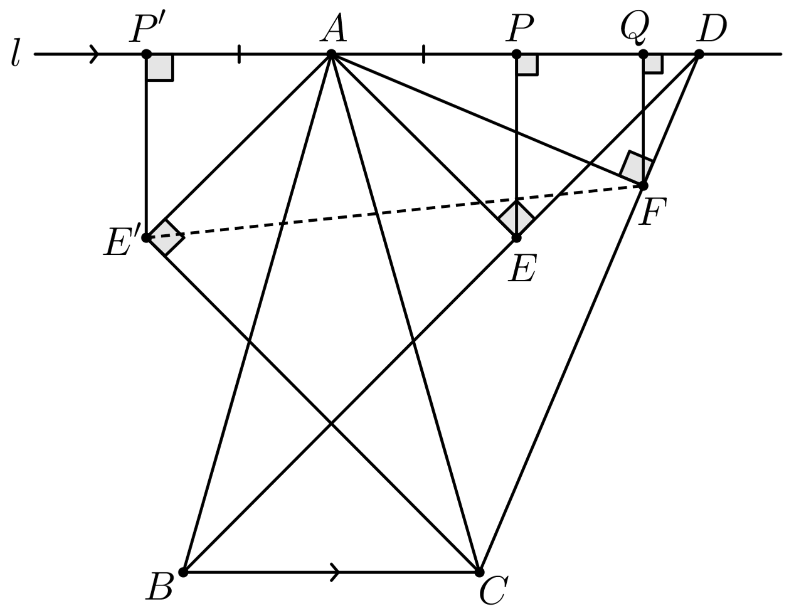
$$\Rightarrow AP + AQ \leq AB \blacksquare$$



Solution2.

Let P', E', be the reflection of P, E with respect to the perpendicular bisector of BC. Therefore, we have $\angle AE'C = \angle AP'E = 90^\circ$ and $AP' = AP$. We have:

$$AP + AQ = AP' + AQ = QP' \leq FE' \leq AC = AB \blacksquare$$



Solutions of 4th Iranian Geometry Olympiad 2017 (Intermediate)

1. Let ABC be an acute-angled triangle with $A = 60°$. Let E, F be the feet of altitudes through B, C respectively. Prove that $CE - BF = \frac{3}{2}(AC - AB)$.

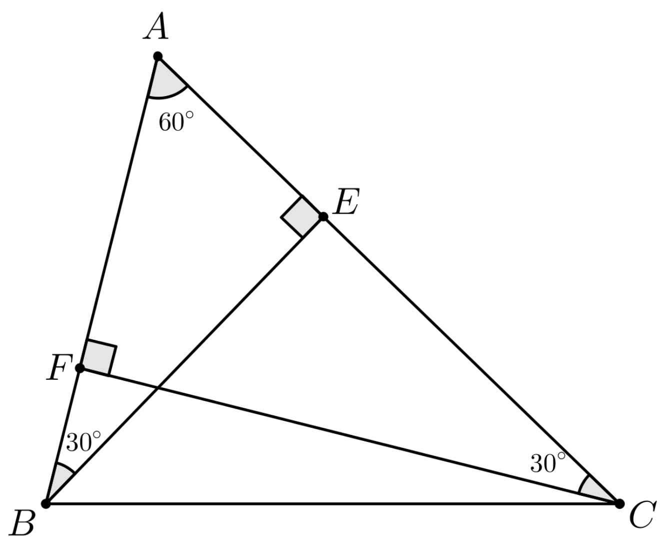
Proposed by Fatemeh Sajadi

Solution.

$$\angle A = 60° \quad \Rightarrow \quad \angle ABF = ACF = 30° \quad \Rightarrow \quad AE = \frac{AB}{2} \quad , \quad AF = \frac{AC}{2}$$

$$\Rightarrow \quad CE - BF = (AC - AE) - (AB - AF)$$

$$= (AC - AB) + (AF - AE) = \frac{3}{2}(AC - AB) \ \blacksquare$$



2. Two circles ω_1, ω_2 intersect at A, B. An arbitrary line through B meets ω_1, ω_2 at C, D respectively. The points E, F are chosen on ω_1, ω_2 respectively so that $CE = CB$, $BD = DF$. Suppose that BF meets ω_1 at P, and BE meets ω_2 at Q. Prove that A, P, Q are collinear.

Proposed by Iman Maghsoudi

Solution.

We know that:

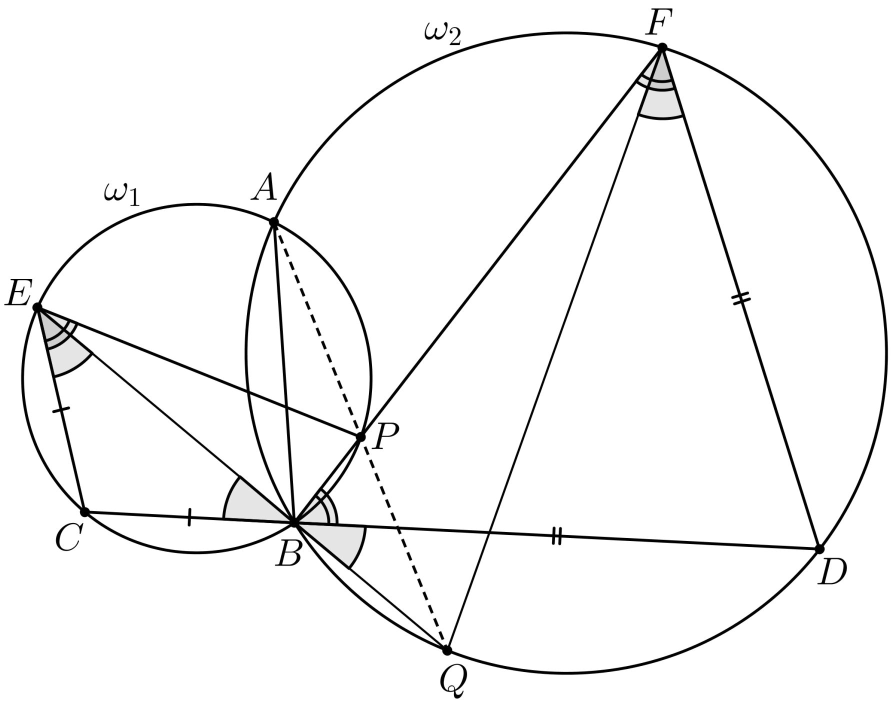
$$\angle BFD = \angle DBF = 180^\circ - \angle CBP = \angle CEP$$

$$\Rightarrow \quad \angle CEB + \angle BEP = \angle BFQ + \angle QFD$$

$$\angle CEB = \angle CBE = \angle QBD = \angle QFD$$

$$\Rightarrow \quad \angle BEP = \angle BFQ \quad \Rightarrow \quad \angle BAP = \angle BEP = \angle BFQ = \angle BAQ$$

So A, P, Q are collinear. ■



3. On the plane, n points are given ($n > 2$). No three of them are collinear. Through each two of them the line is drawn, and among the other given points, the one nearest to this line is marked (in each case this point occurred to be unique). What is the maximal possible number of marked points for each given n?

Proposed by Boris Frenkin (Russia)

— —

Solution.

The case $n = 3$ is obvious. For $n > 4$, take a regular n-gon and slightly deform it to make the assumptions of the problem valid. Each vertex is the nearest to the line connecting two adjacent vertices (this is directly proved by computing angles).

If $n = 4$ then two cases are possible.

1) The given points form a convex quadrilateral (say $ABCD$). Since for each side the nearest vertex is unique, no two sides are parallel. Let lines AD and BC meet beyond A and B, and lines AB and CD meet beyond B and C respectively. Then points A, B and C are marked. Suppose D is marked as well. Then it is the nearest vertex to AC, and the area of $\triangle\ ACD$ is less than that of $\triangle\ ACB$. On the other hand, draw the line through C parallel to AD. Let it meet AB at point E. Then $CE < AD$, so $S_{\triangle\ ACD} > S_{\triangle\ ACE} > S_{\triangle\ ACB}$, a contradiction.

2) One of the given points (say O) is inside the triangle ABC formed by the remaining points. Obviously O is nearest of the given points to lines AB, BC and AC. Without loss of generality, the nearest point to line AO is B. If the nearest point to line BO is A then C cannot be marked. Suppose the nearest point to line BO is C. If the remaining point A is also marked then it is nearest to line CO. Let line AO intersect BC at point A_1, and let points B_1, C_1 be defined similarly. We have $S_{\triangle\ AA_1B} < S_{\triangle\ AA_1C}$, hence $A_1B < A_1C$. Similarly $B_1C < B_1A$, $C_1A < C_1B$. Then $A_1B \cdot B_1C \cdot C_1A < A_1C \cdot B_1A \cdot C_1B$, a contradiction with Ceva theorem. ■

— —

Comment.

In fact, it is not necessary to apply Ceva theorem. Let M_A, M_B, M_C be the bases of the corresponding medians, and M be the centroid. Then O must belong to the interior of each of triangles MM_AB, MM_BC, MM_CA which is impossible.

4. In the isosceles triangle ABC ($AB = AC$), let l be a line parallel to BC through A. Let D be an arbitrary point on l. Let E, F be the feet of perpendiculars through A to BD, CD respectively. Suppose that P, Q are the images of E, F on l. Prove that $AP + AQ \leq AB$.

Proposed by Morteza Saghafian

Solution1.

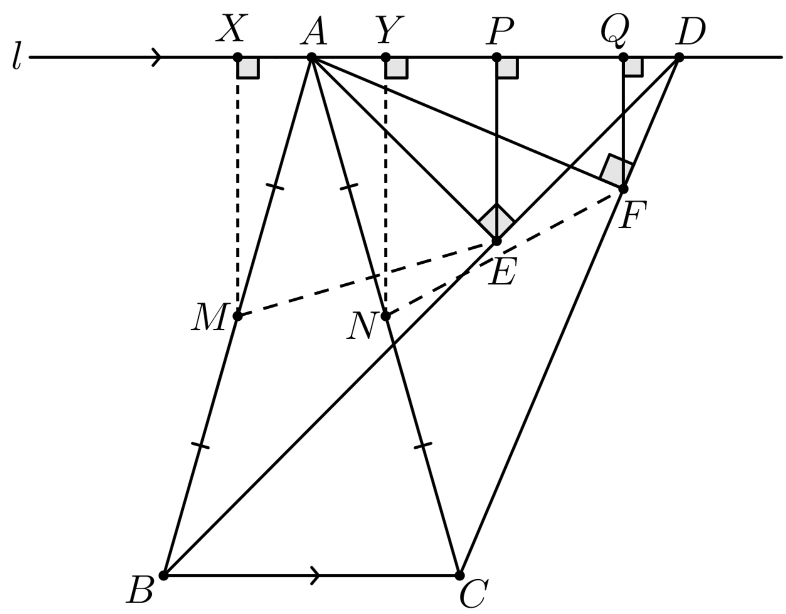
Suppose that M, N are the midpoints of AB , AC respectively. We have:

$$\angle AEB = \angle AFC = 90^\circ \quad \Rightarrow \quad ME = NF = \frac{AB}{2} = \frac{AC}{2}$$

Let X, Y be the feet of perpendiculars through M, N to line l respectively. We know that two triangles AMA and ANY are equal so $AX = AY$.

$$XP = AX + AP \leq ME = \frac{AB}{2} \quad , \quad AQ - AY = YQ \leq NF = \frac{AC}{2}$$

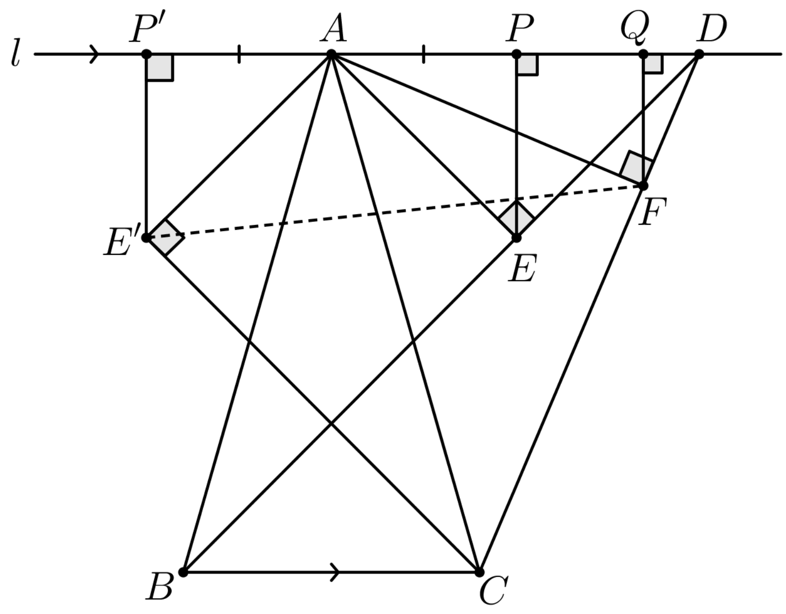
$$\Rightarrow AP + AQ \leq AB \ \blacksquare$$



Solution2.

Let P', E', be the reflection of P, E respect to the perpendicular bisector of BC. Therefore, we have $\angle AE'C = \angle AP'E = 90^\circ$ and $AP' = AP$. We have:

$$AP + AQ = AP' + AQ = QP' \leq FE' \leq AC = AB \blacksquare$$

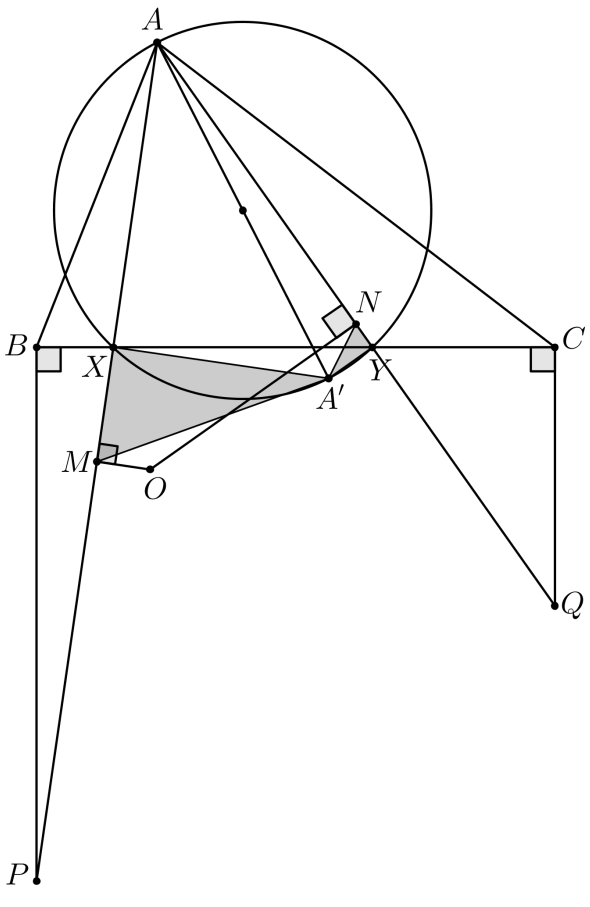


5. Let X, Y be two points on the side BC of triangle ABC such that $2XY = BC$. (X is between B, Y) Let AA' be the diameter of the circumcircle of triangle AXY. Let P be the point where AX meets the perpendicular from B to BC, and Q be the point where AY meets the perpendicular from C to BC. Prove that the tangent line from A' to the circumcircle of AXY passes through the circumcenter of triangle APQ.

Proposed by Iman Maghsoudi

Solution.

Let O, M, N be the circumcenter of triangle APQ, midpoint of AP, midpoint of AQ respectively. We know that $\angle OMA = \angle ONA = 90°$ so the quadrilateral $AMON$ is cyclic. We have to prove $\angle OA'A = 90°$. Therefore, we have to show that $AMOA'N$ is cyclic. It's enought to show that the quadrilateral $AMA'N$ is cyclic. We should prove $\angle A'XM = \angle A'NY$. We will show that $\triangle A'NY \sim \triangle A'XM$.



We know that the quadrilateral $AXA'Y$ is cyclic so $\angle A'XM = \angle A'YN$. Therefore, it's enough to show that:

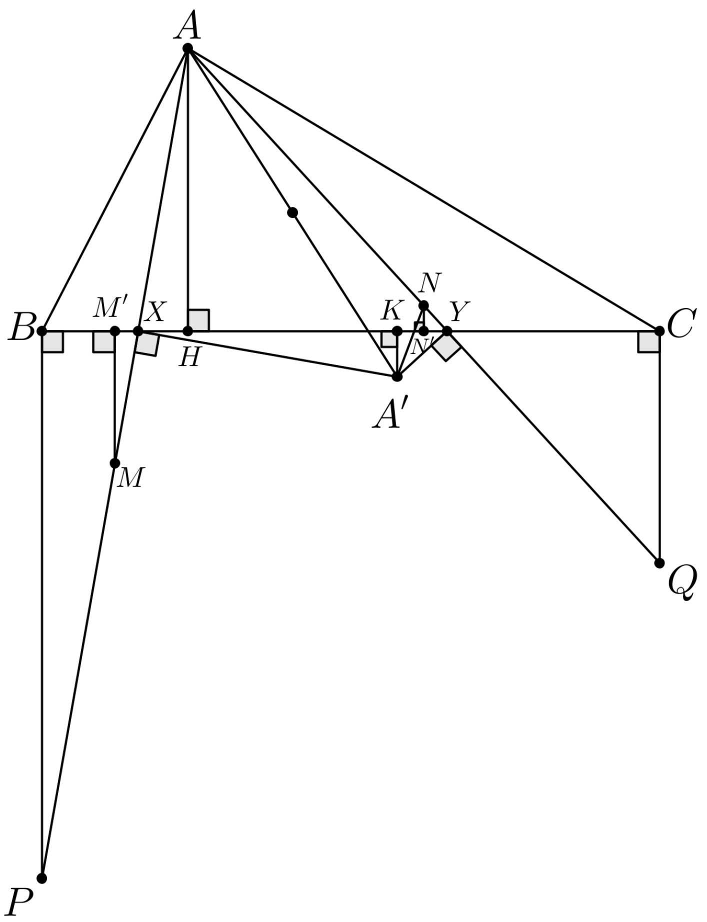
$$\frac{A'X}{MX} = \frac{A'Y}{NY} \quad (1)$$

Let H, K, M', N' be the feet of perpendiculars through A, A', M, N to BC respectively. We know that M, N are the midpoints of AP, AQ so we have:

$$BM' = HM' \quad \Rightarrow \quad XM' = HM' - XH = \frac{BH}{2} - XH \quad (2)$$

$$CN' = HN' \quad \Rightarrow \quad YN' = HY - HN' = HY - \frac{CH}{2} = (XY - XH) - \frac{BC - BH}{2}$$

$$\Rightarrow \quad YN' = (XY - \frac{BC}{2}) + (\frac{BH}{2} - XH) = \frac{BH}{2} - XH \quad (3)$$



According to the equations (2), (3) we can say that $XM' = YN'$. On the other hand, we have AA' is the diameter of the circumcircle of triangle AXY. Therefore:

$$\angle A'YA = \angle A'XA = 90^\circ \quad \Rightarrow \quad \angle KA'Y = \angle NYN' \quad , \quad \angle KA'X = \angle MXM'$$

$$\Rightarrow \quad \triangle A'KX \sim \triangle XM'M \quad , \quad \triangle A'KY \sim \triangle YN'N$$

$$\Rightarrow \quad \frac{A'X}{MX} = \frac{A'K}{XM'} \quad , \quad \frac{A'K}{NY'} = \frac{A'Y}{NY} \quad \Rightarrow \quad \frac{A'X}{MX} = \frac{A'Y}{NY}$$

So equation (1) is proved. ■

Solutions of 4th Iranian Geometry Olympiad 2017 (Advanced)

1. In triangle ABC, the incircle, with center I, touches the side BC at point D. Line DI meets AC at X. The tangent line from X to the incircle (different from AC) intersects AB at Y. If YI and BC intersect at point Z, prove that $AB = BZ$.

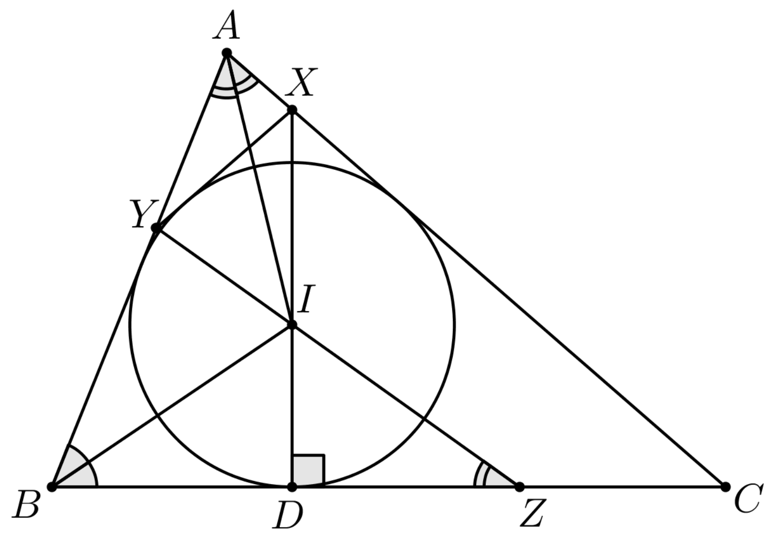
Proposed by Hooman Fattahimoghaddam

Solution.

In triangle AXY, the point I is excenter so $\angle XIY = 90^\circ - \frac{A}{2}$.

$$\Rightarrow \quad \angle DIZ = 90^\circ - \frac{A}{2} \quad \Rightarrow \quad \angle IZB = \frac{A}{2} = \angle BAI$$

Also we know that $\angle IBZ = \angle IBA$ and $BI = BI$ so two triangles ABI and ZBI are equal. Thus, $AB = BZ$. ■



2. We have six pairwise non-intersecting circles that the radius of each is at least one. Prove that the radius of any circle intersecting all the six circles, is at least one.

Proposed by Mohammad Ali Abam – Morteza Saghafian

––

Solution.

Call the centers of these six circles $O_1, O_2, ..., O_6$, and their radii $R_1, R_2, ..., R_6$. Suppose that a circle with center O and radius R intersects these six circles. Obviously there exist i, j such that $O_iOO_j < 60°$. The length of O_iO_j is at least $R_i + R_j$ and the lengths of OO_i, OO_j are less than or equal to $R_i + R, R_j + R$, respectively. If $R < 1$ then considering $R_i, R_j > 1$ we conclude that in triangle O_iOO_j the longest side is O_iO_j, so $O_iOO_j \geq 60°$ which yields a contradiction. So we must have $R \geq 1$. ■

3. Let O be the circumcenter of triangle ABC. Line CO intersects the altitude through A at point K. Let P, M be the midpoints of AK, AC respectively. If PO intersects BC at Y, and the circumcircle of triangle BCM meets AB at X, prove that $BXOY$ is cyclic.

Proposed by Ali Daeinabi – Hamid Pardazi

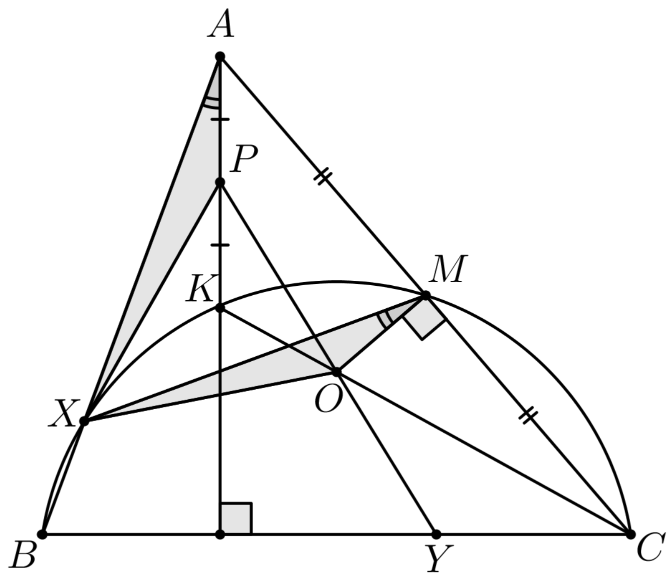
Solution.

We want to show that $\angle XOP = \angle B$. We know that:

$$\angle B = \angle XMA \quad \Rightarrow \quad \angle XMO = 90^\circ - \angle B = \angle XAK$$

We will prove that two triangles XOP and XMA are similar. It's enough to show that two triangles XPA and XOM are similar. We have $\angle XMO = \angle XAK$ so it's enough to prove:

$$\frac{AX}{XM} = \frac{AP}{OM}$$



The quadrilateral $BXMC$ is cyclic. Therefore $\triangle AXM \sim \triangle ACB$.

$$\Rightarrow \frac{AX}{XM} = \frac{AC}{BC} \quad (1)$$

The point O is the circumcenter of triangle ABC. Thus, $\angle OCA = 90^\circ - \angle B$ and $\angle AKC = 180^\circ - \angle A$. According to law of sines in triangles AKC, OMC and ABC we can say that:

$$\frac{AC}{sin\angle A} = \frac{AK}{sin(90^\circ - \angle B)} \quad , \quad OC = \frac{OM}{sin(90^\circ - \angle B)}$$

$$\Rightarrow \quad \frac{AK}{OM} = \frac{AC}{OC.sin\angle A} \quad \Rightarrow \quad \frac{AP}{OM} = \frac{AC}{2OC.sin\angle A} = \frac{AC}{BC} \quad (2)$$

According to equations (1) and (2), we can conclude that two triangles XPA and XOM are similar. Therefore two triangles XOP and XMA are similar so:

$$\angle XBP = \angle XMA = \angle B \ \blacksquare$$

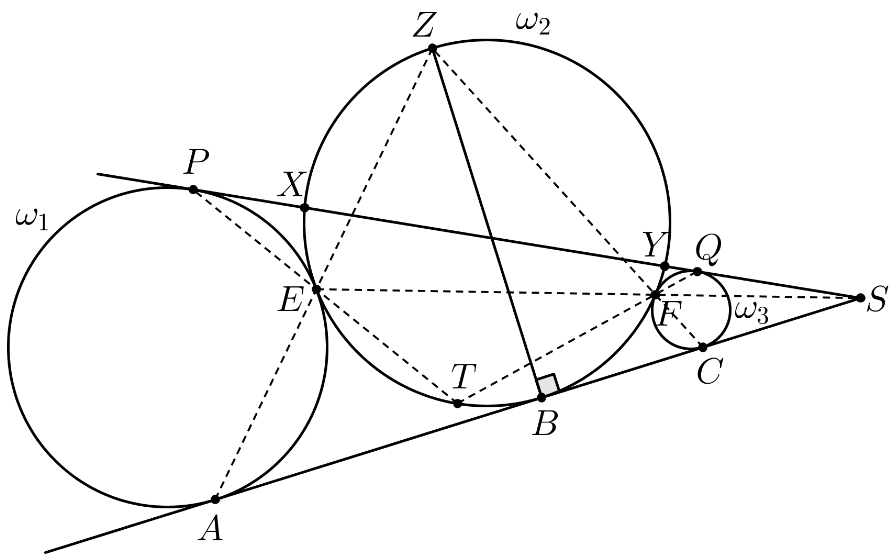
4. Three circles ω_1, ω_2, ω_3 are tangent to line l at points A, B, C (B lies between A, C) and ω_2 is externally tangent to the other two. Let X, Y be the intersection points of ω_2 with the other common external tangent of ω_1, ω_3. The perpendicular line through B to l meets ω_2 again at Z. Prove that the circle with diameter AC touches ZX, ZY.

Proposed by Iman Maghsoudi − Siamak Ahmadpour

Solution.

Let S be the intersection of XY and l. Suppose that ω_1, ω_2 are tangent to each other at point E and ω_2, ω_3 are tangent to each other at point F. We know that S is the external homothetic center for ω_1, ω_3 and E, F are antihomologous points so E, F, S are collinear. Suppose that XY meets ω_1, ω_2 at P, Q respectively so we have:

$$SE \cdot SF = SP \cdot SQ = SA \cdot SC$$



Therefore, the quadrilaterals $PEFQ$ and $AEFC$ are cyclic. Let T be the midpoint of arc XY in circle ω_2 (the arc which has not Z).We know that the line tangent to ω_2 at T is parallel to XY so T, E, P are collinear. Similarly T, Q, F are collinear.

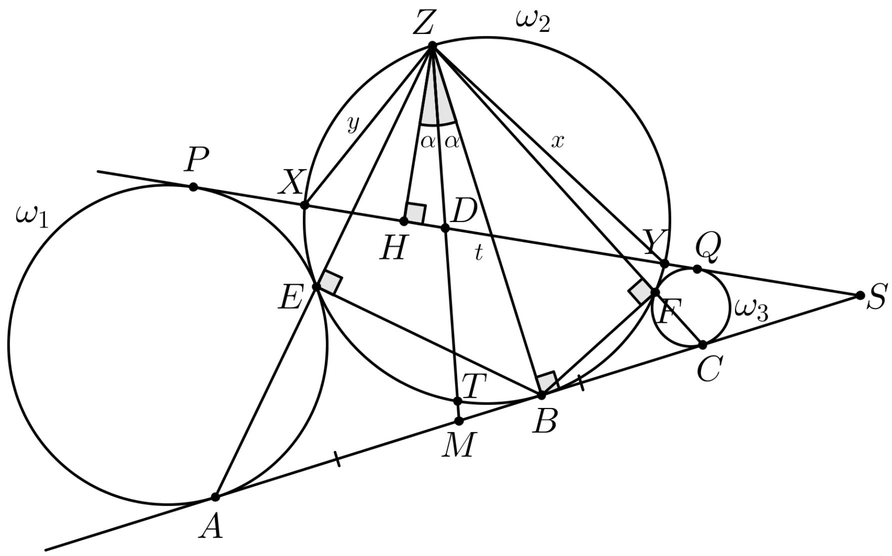
$$TE \cdot TP = TF \cdot TQ \quad \Rightarrow \quad P_{\omega_1}(T) = P_{\omega_3}(T) \quad (1)$$

On the other hand, the line which is tangent to ω_2 at Z is parallel to l so Z, E, A are collinear. Similarly Z, F, C are collinear.

$$ZE \cdot ZA = ZF \cdot ZC \quad \Rightarrow \quad P_{\omega_1}(Z) = P_{\omega_3}(Z) \quad (2)$$

According to the equations (1), (2) we can say that ZT is the radical axis of ω_1, ω_3. Therefore, if M is the midpoint of AC then Z, T, M are collinear because M lies on radical axis of ω_1, ω_3 too. Let D be the intersection of ZM, PQ and H be the feet of perpendiculars through Z to PQ. We know that:

$$\angle HZD = \angle MZB = \frac{|\angle ZXY - \angle ZYX|}{2} = \alpha$$



Let $x = ZY$, $y = ZX$, $t = XY$. We can say that:

$$AM = \frac{ZB}{\cos\alpha} = \frac{ZB}{\frac{ZH}{ZD}} = \frac{ZB}{ZH} \cdot ZD = \frac{ZB}{ZH} \cdot \frac{2xy}{x+y} \cdot \cos\frac{(\angle XZY)}{2}$$

To prove the problem, it's enough to show that the distances of M from two lines ZX, ZY are equal to $\frac{AC}{2}$. Thus, we have to show that:

$$AM \cdot \sin\frac{\angle XZY}{2} = \frac{AC}{2}$$

$$\iff \quad 2AM \cdot \sin\frac{(\angle XZY)}{2} = \frac{ZB}{ZH} \cdot \frac{2xy}{x+y} \cdot \sin(\angle XZY) = AC$$

On the other hand, we have:

$$S_{\triangle XYZ} = \frac{1}{2}t \cdot ZH = \frac{1}{2}xy\sin(\angle XZY) \quad \Rightarrow \quad t = \frac{xy\sin(\angle XZY)}{ZH}$$

So it's enough to show that:

$$AC = \frac{2t \cdot ZB}{x+y}$$

We know that:

$$\angle BFZ = \angle BEZ = 90^\circ \quad \Rightarrow \quad ZB^2 = ZE \cdot ZA = ZF \cdot ZC$$

Therefore, lenghts of tangents from Z to ω_1, ω_3 are equal to ZB. Now, According to Casey's theorem (generalization of Ptolemy's theorem) in two cyclic quadrilaterals $ZY\omega_1X$ and $ZX\omega_3Y$ we can say that:

$$ZY\omega_1X : x \cdot PX + y \cdot (t + PX) = t \cdot ZB \quad \Rightarrow \quad (x+y)PX + yt = t \cdot ZB$$

$$ZX\omega_3Y : y \cdot YQ + x \cdot (t + YQ) = t \cdot ZB \quad \Rightarrow \quad (x+y)YQ + xt = t \cdot ZB$$

$$\Rightarrow \quad PX + YQ + t = \frac{2t \cdot ZB}{x+y} \quad \Rightarrow \quad AC = \frac{2t \cdot ZB}{x+y} \blacksquare$$

5. Sphere S touches a plane. Let A, B, C, D be four points on this plane such that no three of them are collinear. Consider the point A' such that S is tangent to the faces of tetrahedron $A'BCD$. Points B', C', D' are defined similarly. Prove that A', B', C', D' are coplanar and the plane $A'B'C'D'$ touches S.

Proposed by Alexey Zaslavsky (Russia)

Solution.

Let S touch the plane at point P. D is the point of concurrency of the planes passing through AB, BC, CA and touching S (similarly for A', B', C'). Suppose that the plane passing through X, Y and touching S, touches it at P_{xy}. The points P, P_{ab}, P_{ac}, P_{ad} are on a circle W_a, because when we connect them to P, the resulted line touches S. Similarly we can find other triples which are concyclic with P, name the circles with W_b, W_c, W_d. Now use inversion with center P and arbitrary radius, we get 4 lines with three points on each line. Considering the concurrency of four circles at Michels Point we conclude that the circles $P_{ab}P_{bc}P_{ca}$, $P_{ab}P_{bd}P_{da}$, $P_{ac}P_{cd}P_{da}$, $P_{bc}P_{cd}P_{db}$ (call them W'_d, W'_c, W'_b, W'_a) are concurrent on S at a point P'. W'_d is the locus of feet of tangencies from D' to S, so $D'P'$ is tangent to S and D' is on the tangent plane at P' to S. Similarly we conclude that A', B', C' are also on the tangent plane at P' to S. So A', B', C', D' are coplanar and this plane is tangent to S. $\blacksquare$