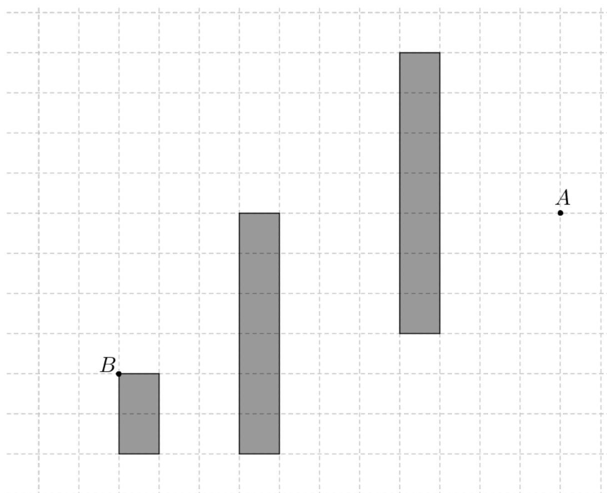
Problems of 3rd Iranian Geometry Olympiad 2016 (Elementary)

1. Ali wants to move from point A to point B. He cannot walk inside the black areas but he is free to move in any direction inside the white areas (not only the grid lines but the whole plane). Help Ali to find the shortest path between A and B. Only draw the path and write its length.



Proposed by Morteza Saghafian

2. Let ω be the circumcircle of triangle ABC with $AC > AB$. Let X be a point on AC and Y be a point on the circle ω, such that $CX = CY = AB$. (The points A and Y lie on different sides of the line BC). The line XY intersects ω for the second time in point P. Show that $PB = PC$.

Proposed by Iman Maghsoudi

3. Suppose that $ABCD$ is a convex quadrilateral with no parallel sides. Make a parallelogram on each two consecutive sides. Show that among these 4 new points, there is only one point inside the quadrilateral $ABCD$.

Proposed by Morteza Saghafian

4. In a right-angled triangle ABC ($\angle A = 90°$), the perpendicular bisector of BC intersects the line AC in K and the perpendicular bisector of BK intersects the line AB in L. If the line CL be the internal bisector of angle C, find all possible values for angles B and C.

Proposed by Mahdi Etesami Fard

5. Let $ABCD$ be a convex quadrilateral with these properties:
$\angle ADC = 135°$ and $\angle ADB - \angle ABD = 2\angle DAB = 4\angle CBD$. If $BC = \sqrt{2}CD$ prove that $AB = BC + AD$.

Proposed by Mahdi Etesami Fard

Problems of 3rd Iranian Geometry Olympiad 2016 (Medium)

1. In trapezoid $ABCD$ with $AB \parallel CD$, ω_1 and ω_2 are two circles with diameters AD and BC, respectively. Let X and Y be two arbitrary points on ω_1 and ω_2, respectively. Show that the length of segment XY is not more than half of the perimeter of $ABCD$.

Proposed by Mahdi Etesami Fard

2. Let two circles C_1 and C_2 intersect in points A and B. The tangent to C_1 at A intersects C_2 in P and the line PB intersects C_1 for the second time in Q (suppose that Q is outside C_2). The tangent to C_2 from Q intersects C_1 and C_2 in C and D, respectively (The points A and D lie on different sides of the line PQ). Show that AD is bisector of the angle CAP.

Proposed by Iman Maghsoudi

3. Find all positive integers N such that there exists a triangle which can be dissected into N similar quadrilaterals.

Proposed by Nikolai Beluhov (Bulgaria) and Morteza Saghafian

4. Let ω be the circumcircle of right-angled triangle ABC ($\angle A = 90°$). Tangent to ω at point A intersects the line BC in point P. Suppose that M is the midpoint of (the smaller) arc AB, and PM intersects ω for the second time in Q. Tangent to ω at point Q intersects AC in K. Prove that $\angle PKC = 90°$.

Proposed by Davood Vakili

5. Let the circles ω and ω' intersect in points A and B. Tangent to circle ω at A intersects ω' in C and tangent to circle ω' at A intersects ω in D. Suppose that the internal bisector of $\angle CAD$ intersects ω and ω' at E and F, respectively, and the external bisector of $\angle CAD$ intersects ω and ω' in X and Y, respectively. Prove that the perpendicular bisector of XY is tangent to the circumcircle of triangle BEF.

Proposed by Mahdi Etesami Fard

Problems of 3rd Iranian Geometry Olympiad 2016 (Advanced)

1. Let the circles ω and ω' intersect in A and B. Tangent to circle ω at A intersects ω' in C and tangent to circle ω' at A intersects ω in D. Suppose that the segment CD intersects ω and ω' in E and F, respectively (assume that E is between F and C). The perpendicular to AC from E intersects ω' in point P and perpendicular to AD from F intersects ω in point Q (The points A, P and Q lie on the same side of the line CD). Prove that the points A, P and Q are collinear.

Proposed by Mahdi Etesami Fard

2. In acute-angled triangle ABC, altitude of A meets BC at D, and M is midpoint of AC. Suppose that X is a point such that $\angle AXB = \angle DXM = 90^\circ$ (assume that X and C lie on opposite sides of the line BM). Show that $\angle XMB = 2\angle MBC$.

Proposed by Davood Vakili

3. Let P be the intersection point of sides AD and BC of a convex qualrilateral $ABCD$. Suppose that I_1 and I_2 are the incenters of triangles PAB and PDC, respectively. Let O be the circumcenter of PAB, and H the orthocenter of PDC. Show that the circumcircles of triangles AI_1B and DHC are tangent together if and only if the circumcircles of triangles AOB and DI_2C are tangent together.

Proposed by Hooman Fattahimoghaddam

4. In a convex quadrilateral $ABCD$, the lines AB and CD meet at point E and the lines AD and BC meet at point F. Let P be the intersection point of diagonals AC and BD. Suppose that ω_1 is a circle passing through D and tangent to AC at P. Also suppose that ω_2 is a circle passing through C and tangent to BD at P. Let X be the intersection point of ω_1 and AD, and Y be the intersection point of ω_2 and BC. Suppose that the circles ω_1 and ω_2 intersect each other in Q for the second time. Prove that the perpendicular from P to the line EF passes through the circumcenter of triangle XQY.

Proposed by Iman Maghsoudi

5. Do there exist six points $X_1, X_2, Y_1, Y_2, Z_1, Z_2$ in the plane such that all of the triangles $X_iY_jZ_k$ are similar for $1 \leq i, j, k \leq 2$?

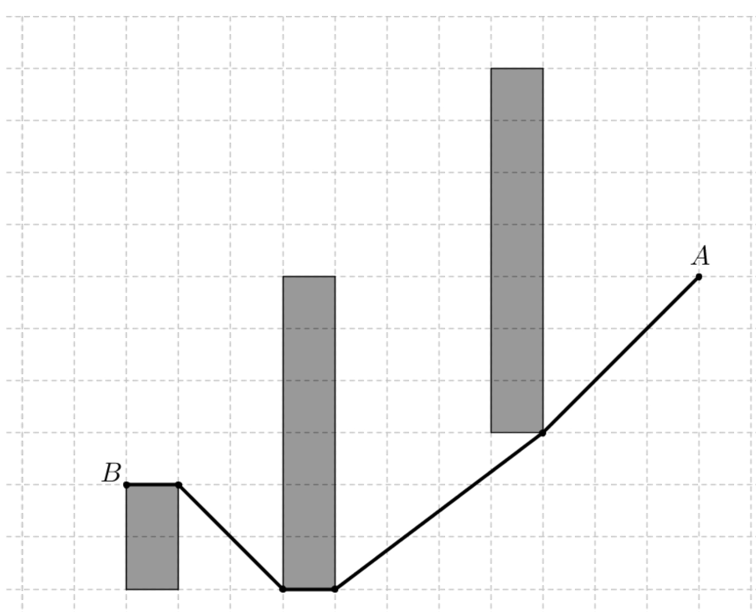
Proposed by Morteza Saghafian

Solutions of 3nd Iranian Geometry Olympiad 2016 (Elementary)

1. Ali wants to move from point A to point B. He cannot walk inside the black areas but he is free to move in any direction inside the white areas (not only the grid lines but the whole plane). Help Ali to find the shortest path between A and B. Only draw the path and write its length.

Proposed by Morteza Saghafian

Solution.



According to Pythagorean theorem, the length of the path AB is equal to:

$$\sqrt{3^2+3^2}+\sqrt{3^2+4^2}+1+\sqrt{2^2+2^2}+1=7+5\sqrt{2}$$

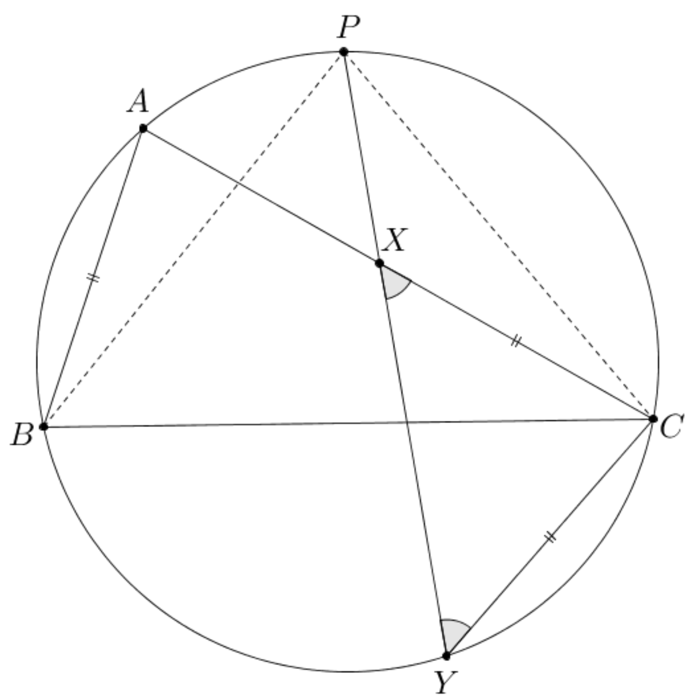
2.Let ω be the circumcircle of triangle ABC with $AC > AB$. Let X be a point on AC and Y be a point on the circle ω, such that $CX = CY = AB$. (The points A and Y lie on different sides of the line BC). The line XY intersects ω for the second time in point P. Show that $PB = PC$.

Proposed by Iman Maghsoudi

Solution.
We know that $CX = CY$ therefore:

$$\angle YXC = \angle XYC \quad \Rightarrow \overset{\frown}{AP} + \overset{\frown}{CY} = \overset{\frown}{PC}$$

Also we have $AB = CY$ therefore $\overset{\frown}{AP} + \overset{\frown}{CY} = \overset{\frown}{AP} + \overset{\frown}{AB} = \overset{\frown}{PB}$, so $PB = PC$.



3. Suppose that $ABCD$ is a convex quadrilateral with no parallel sides. Make a parallelogram on each two consecutive sides. Show that among these 4 new points, there is only one point inside the quadrilateral $ABCD$.

Proposed by Morteza Saghafian

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Solution.
It's clear that the ray from B parallel to AD passes through the quadrilateral if and only if $\angle DAB + \angle ABC > 180°$.

We have to find a parallelogram such that both of it's rays pass thorough $ABCD$. Among A, B and C, D there is exactly one set with sum of angles greater than $180°$. Also among A, D and B, D there is exactly one set with sum of angles greater than $180°$. These two good sets have a vertex in common, say A. So both of the rays from B parallel to AD, and from D parallel to AB, are inside the quadlirateral.

4. In a right-angled triangle ABC ($\angle A = 90°$), the perpendicular bisector of BC intersects the line AC in K and the perpendicular bisector of BK intersects the line AB in L. If the line CL be the internal bisector of angle C, find all possible values for angles B and C.

Proposed by Mahdi Etesami Fard

Solution.
We have three cases:

Case i. $AC > AB$. We know that:

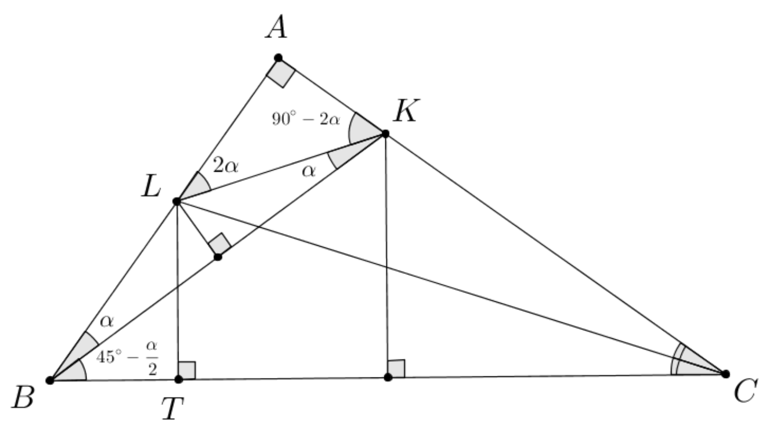
$$\angle LBK = \angle LKB = \alpha \quad \Rightarrow \quad \angle KLA = 2\alpha \quad \Rightarrow \quad \angle LKA = 90° - 2\alpha$$

$$BK = CK \quad \Rightarrow \quad \angle KBC = \angle KCB = \frac{\angle BKA}{2} = 45° - \frac{\alpha}{2}$$

Let T be a point on BC such that $LT \perp BC$. We know that the line CL is the internal bisector of angle C, so $LT = LA$ also we have $LB = LK$ therefore two triangles BTL and KAL are congruent.

$$\Rightarrow \quad \angle LBT = \angle LKA \quad \Rightarrow \quad 45° + \frac{\alpha}{2} = 90° - 2\alpha \quad \Rightarrow \quad \alpha = 18°$$

Therefore $\angle B = 45° + \frac{\alpha}{2} = 54°$ and $\angle C = 36°$



Case ii. $AC < AB$. We know that:

$$\angle LBK = \angle LKB = \alpha \quad \Rightarrow \quad \angle KLA = 2\alpha \quad \Rightarrow \quad \angle LKA = 90^\circ - 2\alpha$$

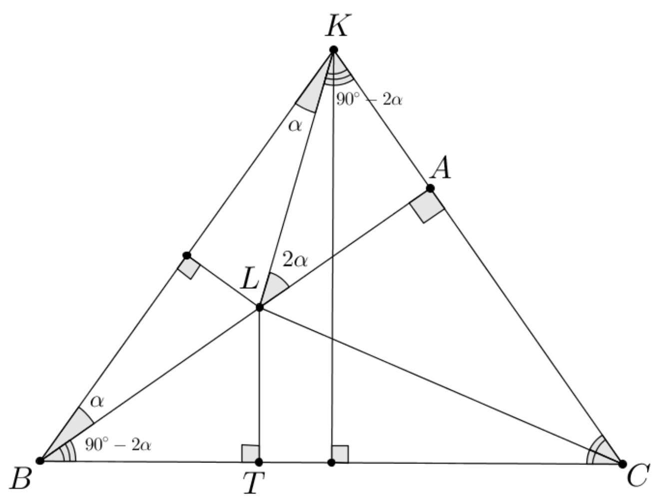
Let T be a point on BC such that $LT \perp BC$. We know that the line CL is the internal bisector of angle C, so $LT = LA$ also we have $LB = LK$ therefore two triangles BTL and KAL are equal.

$$\Rightarrow \angle LBT = \angle LKA = 90^\circ - 2\alpha \Rightarrow \quad \angle CBK = \angle BKC = 90^\circ - \alpha$$

On the other hand we have:

$$BK = CK \quad \Rightarrow \quad \angle CBK = \angle BKC = 60^\circ \Rightarrow \quad \alpha = 30^\circ$$

Therefore $\angle B = 90^\circ - 2\alpha = 30^\circ$ and $\angle C = 60^\circ$



Case iii. $AC = AB$. In this case, $K \equiv A$ and L is the midpoint of AB. Let T be a point on BC such that $LT \perp BC$. We know that the line CL is the internal bisector of angle C, so $LT = LA = LB$ which is impossible.

5. Let $ABCD$ be a convex quadrilateral with these properties:
$\angle ADC = 135^\circ$ and $\angle ADB - \angle ABD = 2\angle DAB = 4\angle CBD$. If $BC = \sqrt{2}CD$ prove that $AB = BC + AD$.

Proposed by Mahdi Etesami Fard

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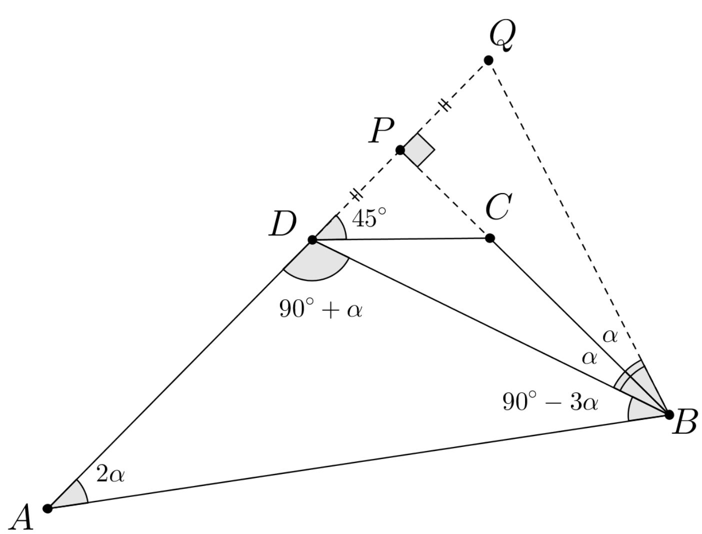
Solution.
Suppose that $\angle CBD = \alpha$, so $\angle DAB = 2\alpha$, therefore:

$$\angle ADB - \angle ABD = 4\alpha \quad , \quad \angle ADB + \angle ABD = 180^\circ - 2\alpha$$

$$\Rightarrow \quad \angle ADB = 90^\circ + \alpha \quad , \quad \angle ABD = 90^\circ - 3\alpha \quad \Rightarrow \quad \angle DAB + \angle CBA = 90^\circ$$

Let P be intersection point of AD and BC. So we have $\angle APB = 90^\circ$. On the other hand we know that $\angle PDC = 45^\circ$, therefore $PD = \frac{\sqrt{2}}{2}CD = \frac{BC}{2}$



Let the point Q be the reflection of point D in point P, Thus $QD = 2PD = BC$. We know that two triangles DPB and QPB are congruent. So $\angle CBD = \angle CBQ = \alpha$, therefore $\angle ABQ = 90^\circ - \alpha$. On the other hand $\angle DAB = 2\alpha$, so the triangle ABQ is isosceles.

$$\Rightarrow \quad AB = AQ \quad \Rightarrow \quad AB = DQ + AD = BC + AD$$

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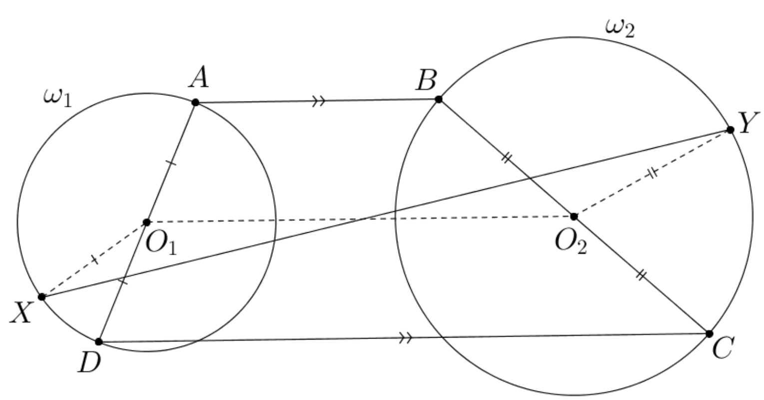
Solutions of 3nd Iranian Geometry Olympiad 2016 (Medium)

1. In trapezoid $ABCD$ with $AB \parallel CD$, ω_1 and ω_2 are two circles with diameters AD and BC, respectively. Let X and Y be two arbitrary points on ω_1 and ω_2, respectively. Show that the length of segment XY is not more than half of the perimeter of $ABCD$.

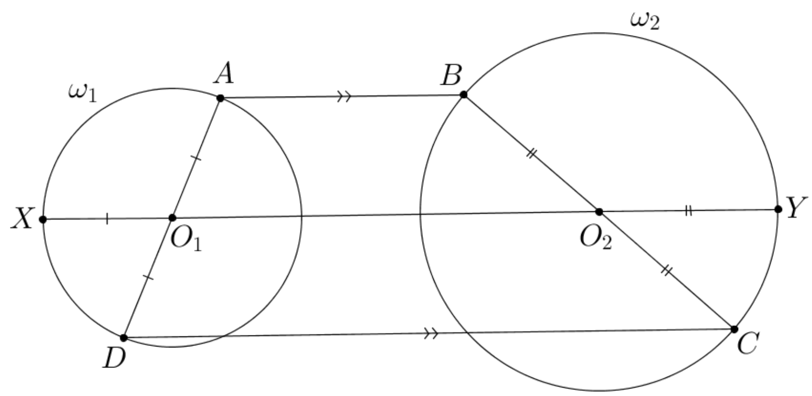
Proposed by Mahdi Etesami Fard

First solution.
Let O_1 and O_2 be the centers of circles ω_1 and ω_2, respectively. It's clear that O_1 and O_2 are the midpoints of AD and BC, respectively.

$$XO_1 = \frac{AD}{2} \quad , \quad YO_2 = \frac{BC}{2} \quad , \quad O_1O_2 = \frac{AB+CD}{2}$$
$$\Rightarrow \quad XY \leq XO_1 + O_1O_2 + YO_2 = \frac{AB+BC+CD+DA}{2}$$



Second solution.
The farthest points of two circles lie on their center line.



And it's clear in the figure that:

$$XO_1 = \frac{AD}{2} \quad , \quad O_1O_2 = \frac{AB + CD}{2} \quad , \quad YO_2 = \frac{BC}{2}$$

2. Let two circles C_1 and C_2 intersect in points A and B. The tangent to C_1 at A intersects C_2 in P and the line PB intersects C_1 for the second time in Q (suppose that Q is outside C_2). The tangent to C_2 from Q intersects C_1 and C_2 in C and D, respectively (The points A and D lie on different sides of the line PQ). Show that AD is bisector of the angle CAP.

Proposed by Iman Maghsoudi

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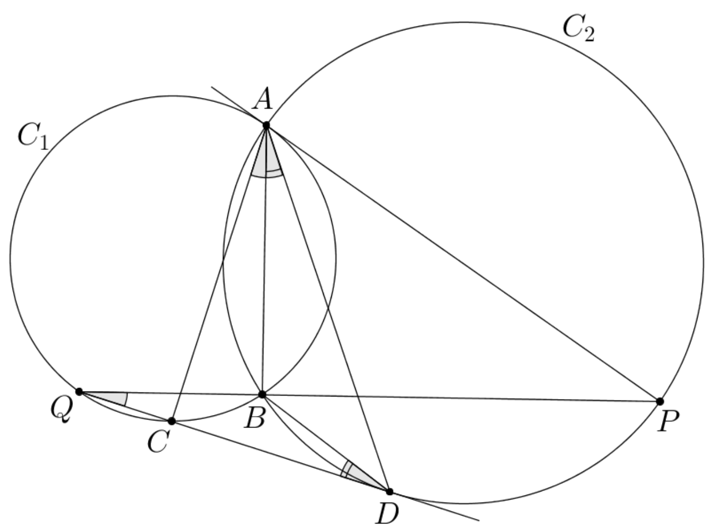
Solution.
We know that:

$$\angle CAB = \angle CQB \quad , \quad \angle DAB = \angle BDQ$$

$$\Rightarrow \quad \angle CAD = \angle CAB + \angle DAB = \angle CQB + \angle BDQ = \angle PBD = \angle PAD$$

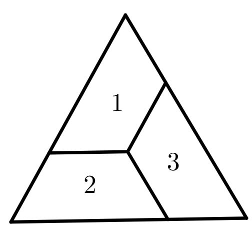
Therefore AD is the bisector of $\angle CAP$.

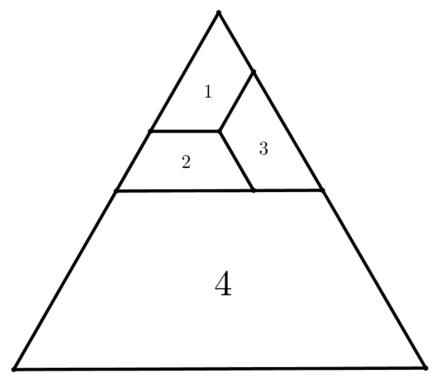


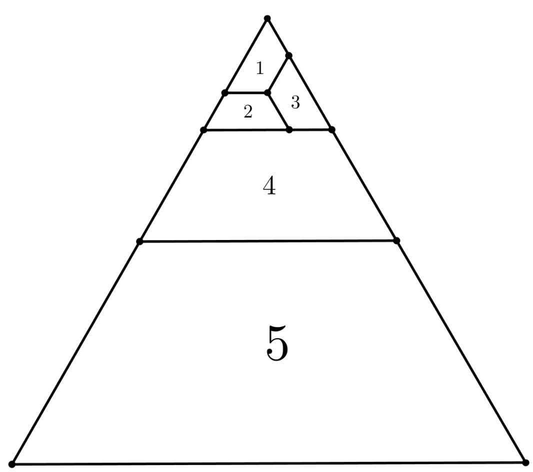
3. Find all positive integers N such that there exists a triangle which can be dissected into N similar quadrilaterals.

Proposed by Nikolai Beluhov (Bulgaria) and Morteza Saghafian

Solution.
For $N = 1$ it's clear that this is impossible. Also for $N = 2$ this dissection is impossible too, because one of the two quadrilaterals is convex and the other is concave. For $N \geq 3$ we can do this kind of dissection in equilateral triangle.







4. Let ω be the circumcircle of right-angled triangle ABC ($\angle A = 90^\circ$). Tangent to ω at point A intersects the line BC in point P. Suppose that M is the midpoint of (the smaller) arc AB, and PM intersects ω for the second time in Q. Tangent to ω at point Q intersects AC in K. Prove that $\angle PKC = 90^\circ$.

Proposed by Davood Vakili

Solution.
Suppose that $AB < AC$. It's enough to show that $PK \parallel AB$.

$$\triangle PMA \sim \triangle PAQ \quad \Rightarrow \quad \frac{AQ}{MA} = \frac{PQ}{PA} \quad , \quad \triangle PMB \sim \triangle PCQ \quad \Rightarrow \quad \frac{MB}{QC} = \frac{PB}{PQ}$$

$$\triangle PBA \sim \triangle PAC \quad \Rightarrow \quad \frac{AC}{BA} = \frac{PA}{PB}$$

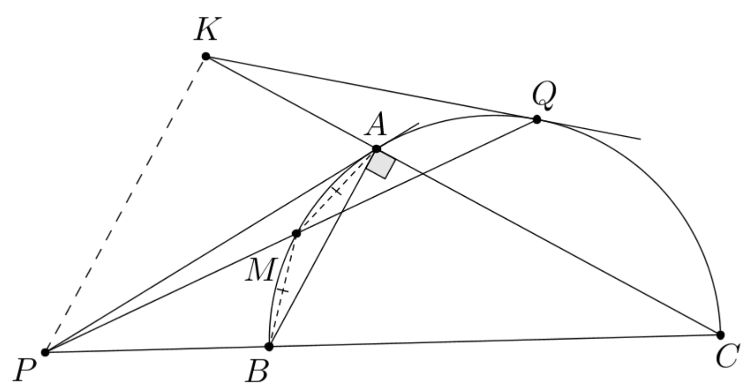
We know that $MA = MB$, so according to above three equations we can say that:

$$\frac{AQ}{QC} = \frac{BA}{AC} \quad (1)$$

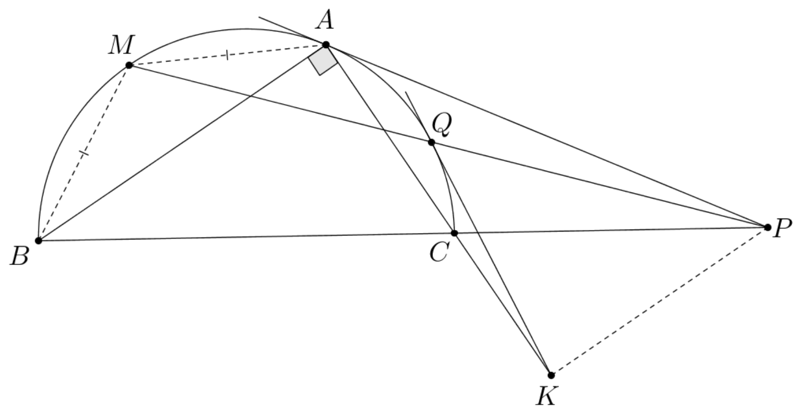
$$\triangle KAQ \sim \triangle KQC \quad \Rightarrow \quad \frac{KA}{KQ} = \frac{KQ}{KC} = \frac{AQ}{QC} \quad \Rightarrow \quad \frac{KA}{KC} = (\frac{AQ}{QC})^2 \quad (2)$$

$$\triangle PBA \sim \triangle PAC \quad \Rightarrow \quad \frac{PB}{PA} = \frac{PA}{PC} = \frac{BA}{AC} \quad \Rightarrow \quad \frac{PB}{PC} = (\frac{BA}{AC})^2 \quad (3)$$

$$(1), (2), (3) \;\Rightarrow\; \frac{KA}{KC} = \frac{PB}{PC} \quad \Rightarrow \quad PK \parallel AB$$



The solution is the same in case of $AB > AC$.



5. Let the circles ω and ω' intersect in points A and B. Tangent to circle ω at A intersects ω' in C and tangent to circle ω' at A intersects ω in D. Suppose that the internal bisector of $\angle CAD$ intersects ω and ω' at E and F, respectively, and the external bisector of $\angle CAD$ intersects ω and ω' in X and Y, respectively. Prove that the perpendicular bisector of XY is tangent to the circumcircle of triangle BEF.

Proposed by Mahdi Etesami Fard

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Solution.
Suppose that P is the intersection point of XE and YF. We know that:

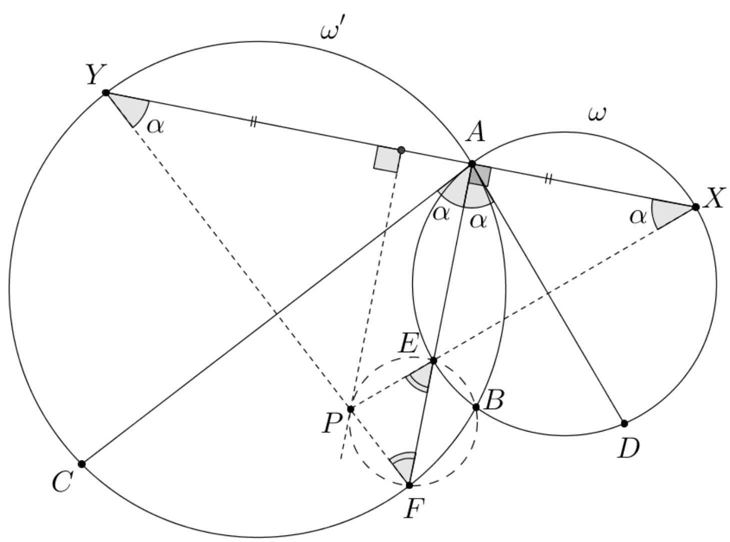
$$\angle EXA = \angle EAC = \angle EAD = \angle FYA = \alpha \quad \Rightarrow \quad PX = PY$$

$$\angle ABE = \angle EXA = \alpha \ \ , \ \ \angle ABF = 180^\circ - \angle FYA = 180^\circ - \alpha$$

$$\Rightarrow \quad \angle EBF = \angle XPY = 180^\circ - 2\alpha \quad \Rightarrow \quad PEBF : cyclic$$

$$EF \perp XY \quad \Rightarrow \quad \angle PEF = \angle AEX = \angle AFY \quad \Rightarrow \quad PE = PF$$

We proved that $PE = PF$ and the quadrilateral $PEBF$ is cyclic. Therefore, P is the midpoint of arc EF in the circumcircle of triangle BEF. Also we know that the perpendicular bisector of XY is parallel to EF and passes through P. So the perpendicular bisector of XY is tangent to the circumcircle of triangle BEF at P.



Solutions of 3nd Iranian Geometry Olympiad 2016 (Advanced)

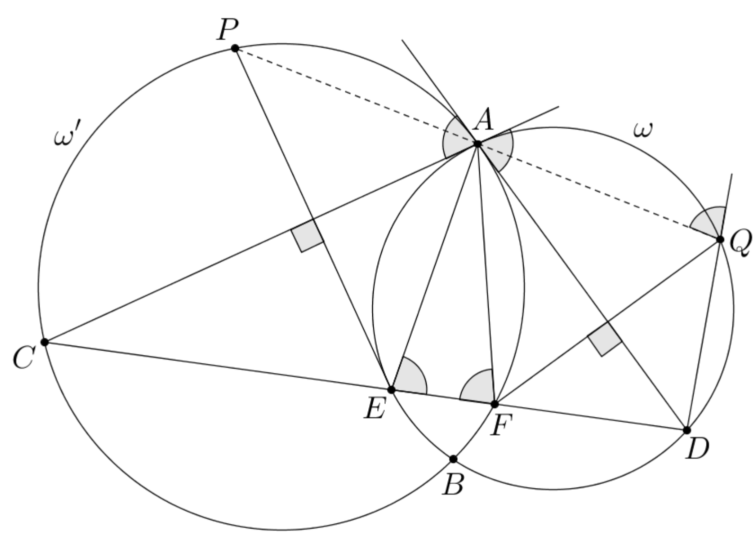
1. Let the circles ω and ω' intersect in A and B. Tangent to circle ω at A intersects ω' in C and tangent to circle ω' at A intersects ω in D. Suppose that CD intersects ω and ω' in E and F, respectively (assume that E is between F and C). The perpendicular to AC from E intersects ω' in point P and perpendicular to AD from F intersects ω in point Q (The points A, P and Q lie on the same side of the line CD). Prove that the points A, P and Q are collinear.

Proposed by Mahdi Etesami Fard

Solution.
We know that:

$$\angle AFC = \angle AED = 180° - \angle CAD \quad , \quad \angle AEF = 180° - \angle AQD$$

$$\Rightarrow \quad \angle AFD = \angle AQD$$



So the point Q is the reflection of the point F in the line AD. Similarly we can say the point P is the reflection of the point E in the line AC. Therefore:

$$\angle DAQ = \angle DAF = \angle ACD \quad , \quad \angle CAP = \angle CAE = \angle CDA$$

$$\Rightarrow \angle DAQ + \angle CAD + \angle CAP = \angle ACD + \angle CAD + \angle CDA = 180°$$

So the points A, P and Q are collinear.

2. In acute-angled triangle ABC, altitude of A meets BC at D, and M is midpoint of AC. Suppose that X is a point such that $\angle AXB = \angle DXM = 90^\circ$ (assume that X and C lie on opposite sides of the line BM). Show that $\angle XMB = 2\angle MBC$.

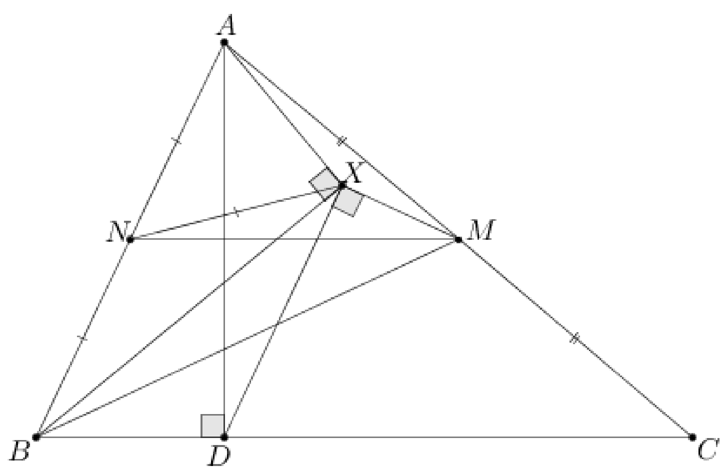
Proposed by Davood Vakili

First solution.
Let N be the midpoint of side AB. So $MN \parallel BC$ and $\angle MBC = \angle NMB$. Therefore it's enough to show that the line MN is the bisector of $\angle XMB$.

$$\angle ADB = \angle AXB = 90^\circ \quad \Rightarrow \quad AXDB : cyclic$$

$$\Rightarrow \quad \angle BXD = \angle BAD = 90^\circ - \angle ABC \quad \Rightarrow \quad \angle BXM = 180^\circ - \angle ABC = \angle BNM$$

$$\Rightarrow \quad BNXM; cyclic \quad , \quad AN = NX = BN \quad \Rightarrow \quad \angle BMN = \angle XMN$$



Second solution.
Let P be the intersection point of XM and BC. Suppose that Q is the point such that the quadrilateral $ADBQ$ be a rectangle. We know that:

$$\angle DXP = \angle ADP = 90^\circ \quad \Rightarrow \quad \angle ADX = \angle XPD$$

Also we know that $AXDBQ$ is cyclic, so:

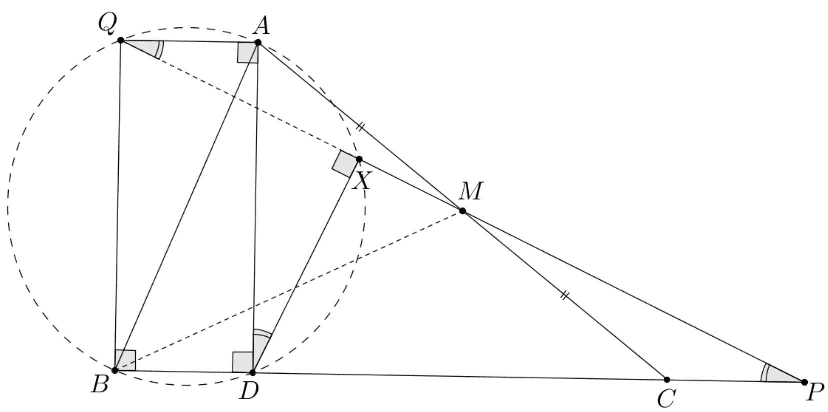
$$\angle ADX = \angle AQX \quad \Rightarrow \quad \angle AQX = \angle XPD$$

So Q, X and P are collinear because $AQ \parallel BP$.

$$AM = MC \ \ and \ \ AQ \parallel BP \quad \Rightarrow \quad QM = MP$$

Now we know that $\angle QBC = 90^\circ$, thus:

$$QM = BM = MP \quad \Rightarrow \quad \angle XMB = 2\angle MBC$$



3. In a convex qualrilateral $ABCD$, let P be the intersection point of AC and BD. Suppose that I_1 and I_2 are the incenters of triangles PAB and PDC respectively. Let O be the circumcenter of PAB, and H the orthocenter of PDC. Show that the circumcircles of triangles AI_1B and DHC are tangent together if and only if the circumcircles of triangles AOB and DI_2C are tangent together.

Proposed by Hooman Fattahimoghaddam

Solution.
Suppose that the circumcircles of triangles AI_1B and DHC is tangent together at point K. Let Q be the second intersection point of circumcircles of triangles AKD and BKC. we know that:

$$\angle DHC = \angle DKC = 180^\circ - \angle P$$

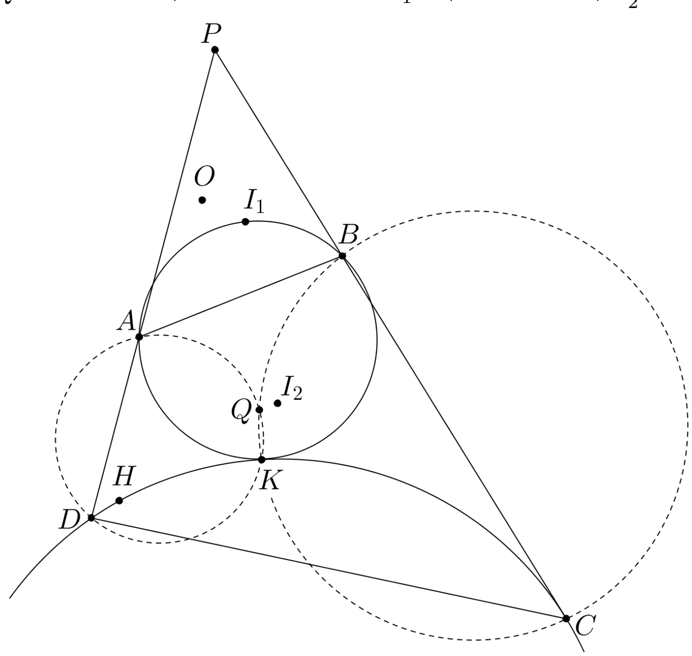
$$\angle P + \angle PDK + \angle PCK = \angle DKC \quad \Rightarrow \quad \angle PDK + \angle PCK = 180^\circ - 2\angle P$$

$$AQKD : cyclic \quad \Rightarrow \quad \angle AQK = 180^\circ - \angle PDK$$

$$BQKC : cyclic \quad \Rightarrow \quad \angle BQK = 180^\circ - \angle PCK$$

$$\Rightarrow \quad \angle AQB = \angle PDK + \angle PCK = 180^\circ - 2\angle P = 180^\circ - \angle AOB \quad \Rightarrow \quad AOBQ : cyclic$$

Also we have $\angle AKD = \angle AQD$, $\angle BKC = \angle BQC$ and $\angle AQB = \angle DKC - \angle P$.
So $\angle CQD = \angle AKB + \angle P = 180^\circ - \angle AI_1B + \angle P = 90^\circ + \frac{\angle P}{2} = \angle CI_2D$.



So the qudrilateral $CDQI_2$ is cyclic. So we have to show that circumcircles of triangles AOB and DI_2C is tangent together at the point Q. It's enough to show that:

$$\angle ABQ + \angle DCQ = \angle AQD$$

We know that the circumcircles of triangles AI_1B and DHC are tangent together at the point K, so we have:

$$\angle ABK + \angle DCK = \angle AKD$$

$$\Rightarrow \quad (\angle ABQ + \angle KBQ) + (\angle DCQ - \angle KCQ) = \angle AKD$$

We know that $\angle KBQ = \angle KCQ$ and $\angle AKD = \angle AQD$, So:

$$\angle ABQ + \angle DCQ = \angle AQD$$

Therefore the circumcircles of triangles AOB and DI_2C are tangent together at point Q.
On the other side of the problem, Suppose that the circumcircles of triangles CI_2D and AOB are tangent together at point Q. Let the point K be the second intersection of circumcircles of triangles AQD and BQC. Similarly we can show that the circumcircles of triangles AI_1B and DHC are tangent together at the point K.

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Comment.
Also there is another solution using inversion with respect to a circle with Michel's point of the quadrilateral as its center.

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4. In a convex quadrilateral $ABCD$, the lines AB and CD meet at point E and the lines AD and BC meet at point F. Let P be the intersection point of diagonals AC and BD. Suppose that ω_1 is a circle passing through D and tangent to AC at P. Also suppose that ω_2 is a circle passing through C and tangent to BD at P. Let X be the intersection point of ω_1 and AD, and Y be the intersection point of ω_2 and BC. Suppose that the circles ω_1 and ω_2 intersect each other in Q for the second time. Prove that the perpendicular from P to the line EF passes through the circumcenter of triangle XQY.

Proposed by Iman Maghsoudi

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First solution.

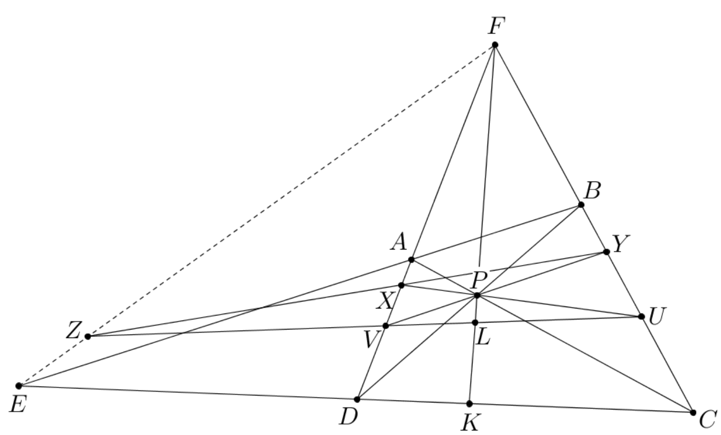
Lemma 1. In the convex quadrilateral $ABCD$, the lines AB and CD meet at point E and the lines AD and BC meet at point F. Let point P be the intersection of AC and BD. Suppose that X and Y be two arbitrary points on AD and BC, respectively. If $BC \cap PX = U$ and $AD \cap PY = V$, then the lines XY, UV and EF are concurrent.

proof.

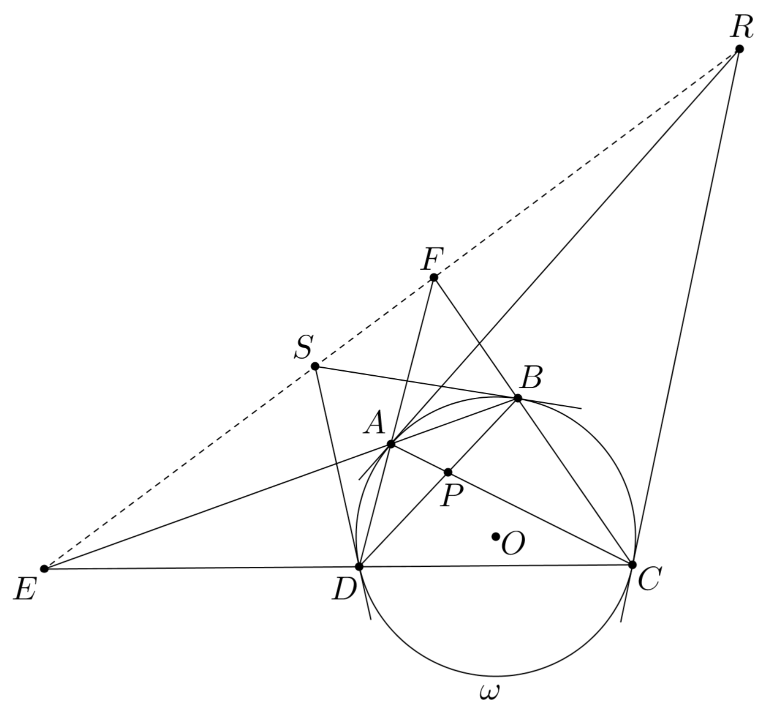
Let point Z be the intersection of XY and UV. Suppose that $PF \cap UV = L$ and $PF \cap CD = K$. We know that:

$$(Z, L, V, U) = -1 \quad , \quad (E, K, D, C) = -1$$

If ZF intersects CD in E', so we can say that $(E', K, D, C) = -1$. Therefore $E \equiv E'$, so the lines XY, UV and EF are concurrent.



Lemma 2. In the cyclic quadrilateral $ABCD$ with circumcenter O, the lines AB and CD meet at point E and the lines AD and BC meet at point F. If point P be the intersection of AC and BD, then $PO \perp EF$.
proof.
Let ω be the circumcircle of quadrilateral $ABCD$. Suppose that point R is the intersection of tangents to circle ω at A and C, and point S is the intersection of tangents to circle ω at B and D.
According to Pascal's theorem in Hexagonal $AABCCD$ and $ABBCDD$, we can say that points R and S lie on line EF.



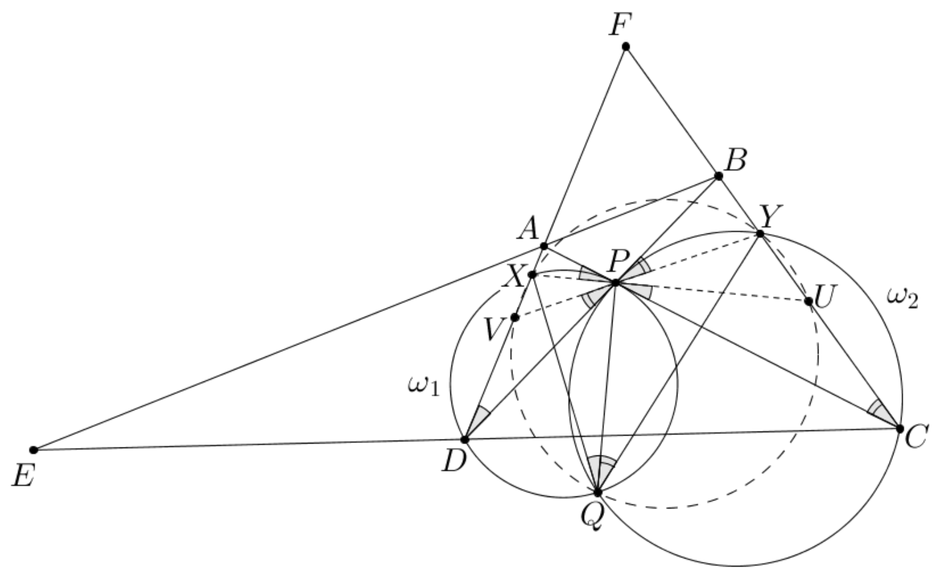
We know that polar of the point R with respect to circle ω passes through P. So polar of the point P with respect to circle ω passes through R. Similarly, we can say that polar of the point P with respect to circle ω passes through S. Therefore polar of the point P with respect to circle ω is EF. So $PO \perp EF$.

Suppose that PX intersects BC in point U, and PY intersects AD in point V.

$$\angle XQP = \alpha \;\;\Rightarrow\;\; \angle XDP = \angle XPA = \angle UPC = \alpha$$

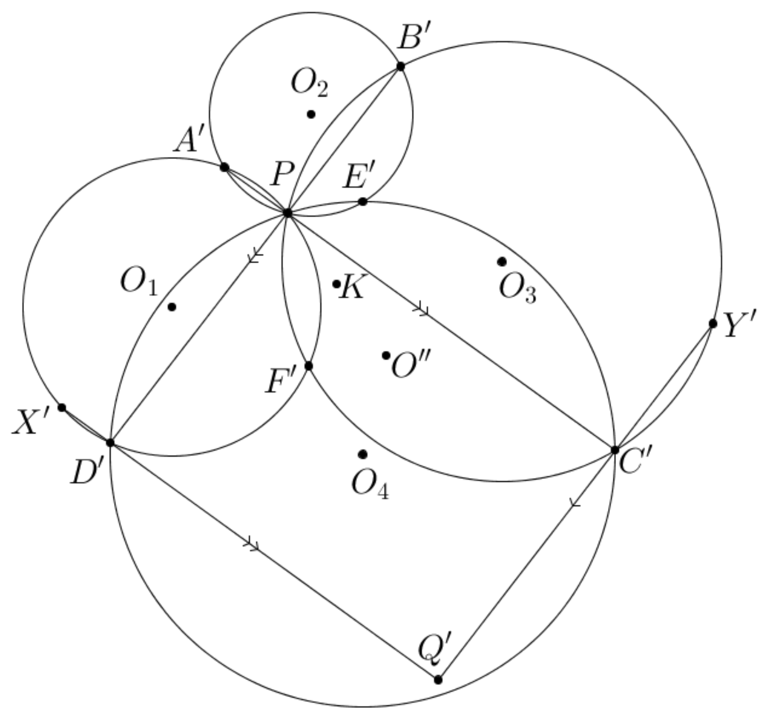
$$\angle YQP = \theta \;\;\Rightarrow\;\; \angle YCP = \angle YPB = \angle VPD = \theta$$

$$\Rightarrow\;\; \angle XVY = \angle XQY = \angle XUY = \alpha + \theta \;\;\Rightarrow\;\; QVXYU : cyclic$$



Let point O be the circumcenter of $QVXYU$. According to lemma 1, we can say that XY, UV and EF are concurrent at point Z. Now according to lamme 2, we can say that $PO \perp EF$. So the perpendicular from P to EF passes through the circumcircle of triangle XQY.

Second solution.
Suppose that point O is the circumcenter of triangle XQY. The inversion with respect to a circle with center P trasnforms the problem into this figure. Suppose that X' is the inversion of point X wrt P. We have to show that the line PO' is the diameter of circumcircle of triangle $E'PF'$. Let O'' be the circumcenter of triangle $X'Q'Y'$. We know that the points P, O' and O'' are collinear. So we have to show that he line PO'' passes through the circumcenter of triangle $E'PF'$.

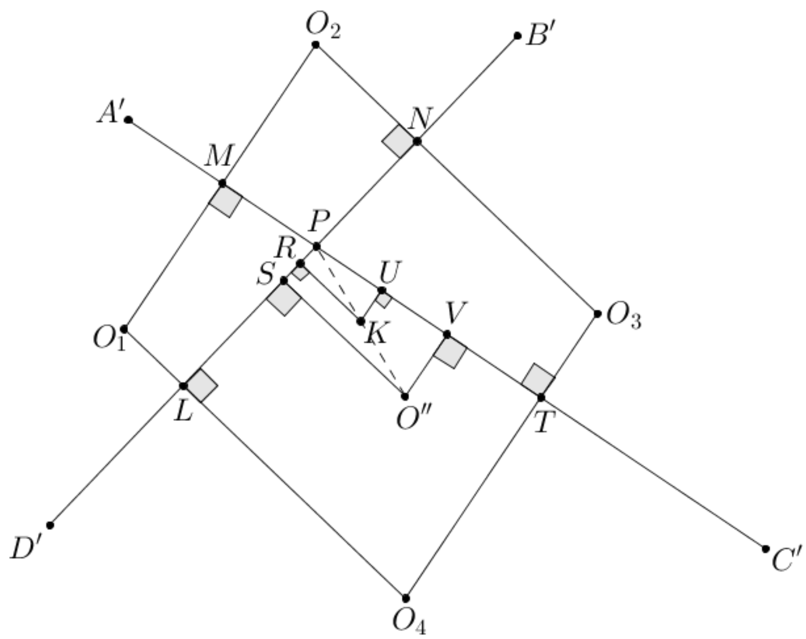


Suppose that O_1, O_2, O_3 and O_4 are the centers of circles in the above figure and K be the intersection point of O_1O_3 and O_2O_4. We know that point K lies on perpendicular bisector of PE' and PF', thus K is the circumcenter of triangle $PE'F'$. So we have to show that P, K and O'' are collinear. On the other hand, we know that the quadrilateral $D'B'Y'Q'$ is isosceles trapezoid. So the point O'' lies on perpendicular bisector of $B'D'$. Similarly, the point O'' lies on perpendicular bisector of $A'C'$. Therefore, the point O'' is the intersection of $A'C'$ and $B'D'$.

Suppose that:

$$A'C' \cap O_1O_2 = M \quad , \quad A'C' \cap O_3O_4 = T$$
$$B'D' \cap O_2O_3 = N \quad , \quad B'D' \cap O_1O_4 = L$$

Let points U and V be on $A'C'$ such that $KU \perp A'C'$ and $O''V \perp A'C'$. Also let points R and S be on $B'D'$ such that $KR \perp B'D'$ and $O''S \perp B'D'$.



We know that O_1O_2 and O_3O_4 are perpendicular to $A'C'$. So $O_1O_2 \parallel O_3O_4$ Similarly $O_2O_3 \parallel O_1O_4$, therefore the quadrilateral $O_1O_2O_3O_4$ is a parallelogram. It means that the point K lies on the midpoint of the segments O_1O_3 and O_2O_4. So $UM = UT$. Also we have $A'M = PM$ and $C'T = PT$

$$\Rightarrow \quad PV = A'V - A'P = (PM + PT) - 2PM = PT - PM$$

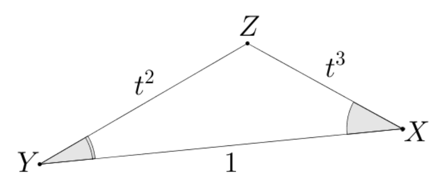
$$\Rightarrow \quad TV = PT - PV = PM \quad \Rightarrow \quad UP = UV$$

Similarly, we can show that $RP = RS$, so point K lies on the perpendicular bisector of PV and PS. It means that K is the citcumcenter of triangle PSV. Therefore the points P, K and O'' are collinear.

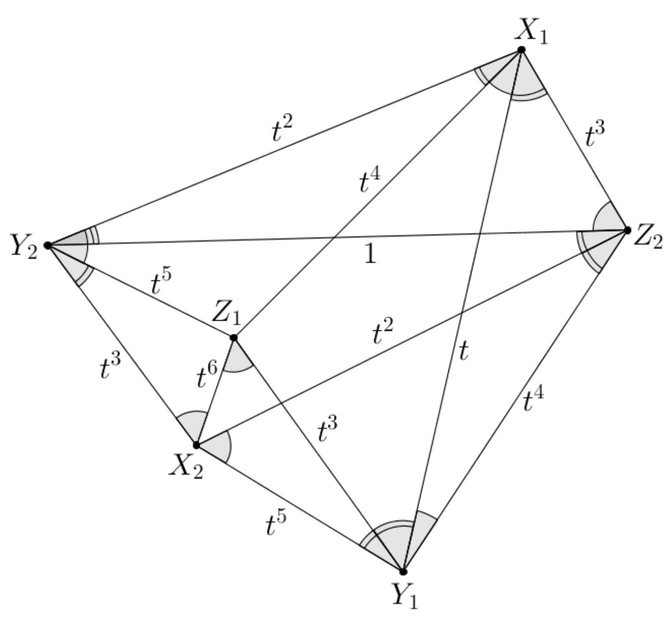
5. Do there exist six points $X_1, X_2, Y_1, Y_2, Z_1, Z_2$ in the plane such that all of the triangles $X_iY_jZ_k$ are similar for $1 \leq i, j, k \leq 2$.

Proposed by Morteza Saghafian

Solution. (*by Ilya Bogdanov from Russia*)
Suppose a triangle XYZ, in such a way that $XY = 1$, $YZ = t^2$, $ZX = t^3$ and $\angle Z = \angle X + 2\angle Y$.



Such a triangle exists, because for the minimum possible value of t, we have $\angle Z > \angle X + 2\angle Y$ and for $t = 1$ we have $\angle Z < \angle X + 2\angle Y$. So there exists a triangle with the above properties. Now consider the following 6 points, these points have the properties of the problem.



So there exist the points $X_1, X_2, Y_1, Y_2, Z_1, Z_2$ in the plane such that $X_i Y_j Z_k$ be the similar triangles for all of $1 \leq i, j, k \leq 2$